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Optimal Management of Durable Pollution

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Abstract

This paper studies an incompletely informed regulator's problem of inducing a firm producing durable pollution to adopt a socially optimal pollution storage technology. We construct a sparse, yet flexible, theoretical model that can be applied directly to concrete situations as it is stated explicitly in terms of statistical parameters. Next, we show the existence of an optimal regulatory contract and examine its qualitative features for one of the many regulatory problems suggested by the general model. Our model extends the standard inventory model by making the firm's storage technology a strategic choice induced by the regulatory contract. Moreover, by providing a structural model of the firm, our model generates useful information that has to be assumed in an abstract regulatory model.

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1. Introduction

1.1 Modelling durability

In this paper we set up and solve the regulator's optimal contracting problem when faced with a firm producing durable pollution. Once created, a durable pollutant survives for some length of time before decaying, thus forcing society to manage the resulting stock over time. As our model accommodates any finite autonomous rate of decay of pollutants, the only pollutants left out of its ambit are those with an infinite autonomous rate of decay, i.e., those that are purely evanescent and dissipate instantaneously.

At any instant, an existing unit of durable pollution can either be managed by the firm or emitted into the public stock (the "environment"). We say that a unit of durable pollution is managed by the firm if it does so internally or delegates this function to an agent who is paid by the firm for this task. As long as a unit of pollution is under the management of the firm or its agent, it is said to be in the private stock. We postulate that only the public stock of pollution is a matter of social concern as it creates negative externalities.

The significant distinction between the two courses of action open to the firm at any instant is that a unit of pollution in the private stock imposes a persistent cost over its lifetime while emitting it implies a one-time cost. This cost structure implies that the firm faces a dynamic optimization problem whose solution at every instant depends on the past *via* the inherited private pollution stock and expectations about the future evolution of the stock. Moreover, if the relevant insurance markets and contracts are incomplete, then the firm faces residual uncertainty regarding many of the factors that determine its operational environment and production of pollution; e.g., the quality of delivered inputs, the efficacy of pollution treatment technology, production fluctuations in response to market conditions, etc. We represent these uncertainties by specifying an explicitly stochastic operational environment for the firm. In addition, the regulator's information about this stochastic environment may be inferior to that of the firm. We capture this asymmetry of information by endowing the firm with private information regarding the parameters determining the firm's stochastic operational environment.

Our model of the firm's environment is explicitly reduced-form and statistical in nature, with the firm seen as a black-box that produces and manages its pollution stock. One reason for adopting this approach is that it allows us to focus sharply on the only aspect of the firm that is relevant to the pollution regulator. Indeed, our statistical modelling of the firm's pollution-related activities is intended to describe a regulator's quantitative perception of a polluting firm. Apart from its descriptive simplicity, our direct approach has the merit of being analytically tractable for many versions of the regulator's problem. Finally, as the model is stated in terms of statistical parameters that can be estimated routinely, we expect that its solution can be calibrated, perturbed, tested and applied quite routinely.

1.2 The regulation problem

The durable pollution created by the firm flows into its private pollution stock.[1] Given a unit of pollution in the private stock, the firm may continue to hold it, entailing an instantaneous holding cost h, or emit it into the public stock, entailing an emission penalty l. The firm pays the holding cost on a unit of pollution as long as it stays in the private stock, while the emission penalty is a one-time charge that permanently relieves the firm of the responsibility of managing that unit. l is given exogenously and is interpreted as the social cost entailed by a unit of pollution in the public stock.

h represents the storage technology chosen by the firm for its private pollution stock. In order to acquire technology h, the firm must invest $\psi(h)$ up-front, where ψ is a decreasing function. Thus, the firm can lower the holding cost it bears on each unit of pollution in its private stock by investing more in storage technology.

We restrict attention to the trade-off between private management and emission by assuming away the possibility of adjusting the firm's production plan.[2] This independence of the firm's production activity from its pollution management activity is implicit in our assumption that the random evolution of the firm's private pollution stock is governed by

[1] For example, holding tanks for liquid wastes, fly-ash dumps at thermal power plants, and stocks of spent fuel at nuclear facilities.

[2] For instance, supply might be governed by inflexible long-term contracts, or the firm might have a linear technology and the cost of managing pollution might be insignificant compared to the profitability of the firm's private good, so that the firm might have no reason to adjust its production level in response to the regulatory contract.

exogenously given parameters x, μ and σ. Let $v(x,\mu,\sigma,\alpha)$ be the exogenously given value of the firm's operations in the private good market, let $V(x,\mu,\sigma)$ be the exogenously given value of the firm to consumers, and let $C(x,\mu,\sigma,h,0,l,\alpha)$ be the firm's cost of optimally managing its private pollution stock, where α is the firm's infinite horizon discount rate.[3]

$x \in \Re_+$ is the firm's type (or private information). As we assume that α, μ, σ and l are exogenously given and common knowledge, we suppress these parameters and denote firm x's market value prior to regulation by $v(x)$, its value to consumers by $V(x)$, the cost of optimally managing its private pollution stock by

$$c(x,h) = C(x,\mu,\sigma,h,0,l,\alpha) \tag{1.2.1}$$

and its utility from outcome (h,T) by

$$u(x,h,T) = v(x) - c(x,h) - \psi(h) + T \tag{1.2.2}$$

where T is a transfer from the regulator to the firm. A pair $\langle \Re_+, (h,T) \rangle$ is called a direct mechanism, with message space $\Re_+$ that coincides with the type space and outcome function $(h,T) : \Re_+ \to \Re_+ \times \Re$; we also refer to outcome functions as contracts. We restrict attention to direct mechanisms without loss of generality because of the revelation principle.[4] Given $\langle \Re_+, (h,T) \rangle$, firm x's utility from reporting type x' is $U(x,x') = u(x, h(x'), T(x'))$.

We consider mechanisms that induce participation and self-selection by all types. The individual rationality (IR) and incentive compatibility (IC) constraints that characterize such contracts are: for all x, $x' \in \Re_+$

$$U(x,x) \geq 0 \qquad \text{and} \qquad U(x,x) \geq U(x,x') \tag{1.2.3}$$

[3] The significance and interpretation of the argument 0 will become clear in Section 2.2.

[4] This basic principle of mechanism design theory (Fudenberg and Tirole 1991) may be stated informally as follows: given the type-contingent outcomes resulting from an equilibrium of the game generated by an arbitrary mechanism, there exists a direct mechanism, i.e., one in which the message spaces coincide with the type spaces, such that (a) truth-telling is an equilibrium of the game generated by this direct mechanism, and (b) the type-contingent outcomes resulting from the truth-telling equilibrium replicate the given type-contingent outcomes. This simplifies the search for an optimal mechanism in two ways. First, one needs to optimize only over the class of direct mechanisms, indeed, only over the class of (direct) outcome functions as the message spaces coincide with the type spaces. Secondly, the restriction to a truth-telling equilibrium allows incentive constraints to be specified simply as the self-selection constraints.

The regulator's welfare is

$$\begin{aligned} W(x,h,T) &= V(x) + u(x,h,T) - (1+\lambda)T \\ &= V(x) + v(x) - c(x,h) - \psi(h) - \lambda T \end{aligned} \tag{1.2.4}$$

where $1+\lambda$, with $\lambda > 0$, is the social shadow value *per* unit of payment by the regulator to the firm. The regulator's problem is to find $(h,T) : \Re_+ \to \Re_+ \times \Re$ to maximize the expectation of (1.2.4), subject to constraints (1.2.3).

It will turn out in our model that a firm's emission activity is positively related to its chosen h. The regulator's interest in the firm's choice of h stems from the following trade-off implied by the negative slope of ψ and the positive association between h and the firm's emissions: high investment $\psi(h)$ in storage technology implies a low holding cost h and low emissions, thereby implying a low social cost caused by emissions (captured by the emission penalty paid by the firm), while low investment $\psi(h)$ in storage technology implies a high holding cost h and high emissions, thereby implying a high social cost caused by emissions. Since the firm's investment cost and holding cost, as well as the social cost of emissions, enter the regulator's welfare function, the regulator needs to compute the optimal trade-off subject to the implementability constraints imposed by information asymmetries and incentive requirements.

The regulatory framework described above modifies a simple Pigovian tax scheme. We show in Section 3 that, if the regulator's information about the firm is incomplete, then optimal regulation requires that the Pigovian tax be supplemented by a transfer scheme designed to induce the firm to adopt the socially desirable storage technology.

1.3 The literature

This paper is concerned with the problem of regulating "point-source stock pollution" (Xepapadeas 1997), i.e., the pollutant is durable and the emitter's identity and the quantity of emissions are perfectly observable. Our approach to this problem departs from the dynamic emission choice model (DECM) in a number of directions.[5] Unfortunately, while

[5] The DECM presented in Chapter 3 of Xepapadeas 1997 is representative of many similar models, including those in Brock 1977, d'Arge and Kogiku 1973, Forster 1973, Keeler et al. 1971, Mäler 1974, Plourde 1972, and Xepapadeas 1992.

the DECM is couched in the capital theoretic formalism of optimal growth models, the substantive features of our approach require us to employ the game theoretic formalism of optimal regulation theory (Laffont and Tirole 1994). This means that our model and the DECM are formally non-comparable and non-nested. Nevertheless, it is possible to understand the substantive differences by considering the following abstract representation of the DECM.

Consider a regulator facing a firm producing durable pollution. Let the public stock of pollution be the state variable and the emission rate the control variable. Given the control trajectory η and the state trajectory S, the regulator's welfare is $w(\eta, S, x)$, where x is a parameter describing the firm. The law of motion for the state imposes a dynamic constraint in the form of a differential equation, say $L(\eta, S) = 0$. The regulator's optimal control trajectory $\eta(x)$ maximizes $w(\eta, S, x)$ subject to the constraint $L(\eta, S) = 0$. With appropriate specifications, as in Xepapadeas 1997, this is an optimal control problem.

The substantive question with respect to this model is whether the regulator can induce firm x to choose the regulator's preferred control $\eta(x)$? Given standard specifications, the regulator can decentralize $\eta(x)$ by imposing a Pigovian tax trajectory $\tau(x)$, whose value at time t is the social shadow cost of the pollution stock at time t. This implementation of $\eta(x)$ relies on the following substantive features of the DECM: (a) x is common knowledge, (b) all produced pollution is emitted, (c) the firm's decision problem at each instant is static, and (d) the firm is a price-taker. Our model differs from the DECM in all these aspects.

First, unlike the DECM, our model features incomplete information, with x as the firm's private information. Consequently, it is impossible for the regulator to implement the socially optimal emissions trajectory $\eta(x)$ *via* Pigovian taxes that are conditioned on x. Secondly, feature (b) of the DECM implies that the entire output of pollution causes a negative externality, and therefore is a matter of social and regulatory concern. Our model introduces an alternative to socially undesirable emission in the form of the possibility of the firm holding the pollution. Thirdly, the choice between holding pollution and emission, with their different cost implications, confronts the firm with a dynamic decision problem at every moment.

As will be shown in Section 2, the firm's emission activity is positively related to the holding cost h. Therefore, the firm in our model will behave like the firm in the DECM by emitting almost all the pollution it creates if the holding cost is high relative to the emission penalty. Thus, the firm's *postulated behavior* in the DECM can emerge endogenously in our model as the firm's rational response to particular regulatory incentives and economic circumstances. Apart from the practical applicability of the model and our enrichment of the description of a polluting firm's environment by placing it in a dynamic stochastic context, we see our enrichment of the regulatory setting (e.g., introduction of incomplete information) as the major departure of our model from the DECM.

1.4 Plan of paper and results

Section 2 of this paper answers the following question: given the firm's operational environment and its pollution processing technology, what is the optimal policy for the firm with respect to the decision whether to emit pollution or to process it internally? We follow Harrison and Taylor 1978 in deriving the answer, which is stated as Theorem 2.3.21. The methodology is to construct a stochastic dynamic programming problem whose solution yields the firm's optimal policy and cost as functions of technological and regulatory parameters.

Section 3 characterizes the optimal regulatory contracts subject to implementability constraints. The cost function derived in Section 2 is the key to the definition and analysis of the incentive constraints in this section. We consider two contracting problems, one with a finite number of types and the other with a continuum of types. The optimal contract for the former problem is stated in Theorem 3.3.13, while the solution of the latter problem is contained in Theorem 3.4.14.

Section 4 concludes the paper with suggestions for extensions of the work reported in this paper.

2. The firm's cost function

2.1 Formal setting

In this section we introduce the formalism and notation that will be used throughout this paper. $\mathcal{Z}_+$ (resp. $\mathcal{Z}_{++}$) denotes the set of nonnegative (resp. positive) integers, $\Re$ (resp. $\Re_+$, $\Re_{++}$) the set of real (resp. nonnegative real, positive real) numbers, D and D^2 are the first order and second order differential operators respectively, Δ denotes a jump of a real-valued variable, and $\langle ., . \rangle$ denotes the predictable quadratic variation process (Elliott 1982, Chapter 10). t denotes an instant of time; unless otherwise specified, $t \in \Re_+$.

Let Ω be the set of continuous real-valued functions with domain $\Re_+$. The Wiener process $W = (W_t)$ is the coordinate process on the stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$, where $(\mathcal{F}_t)$ is a filtration on Ω, $\sigma(\bigcup_{t\in\Re_+} \mathcal{F}_t) \subset \mathcal{F}$, and Q is the unique (Wiener) measure on $(\Omega, \mathcal{F})$ under which W is a Wiener process with zero drift, unit variance, and starting state 0 Q-a.s. We assume, without loss of generality, that $(\mathcal{F}_t)$ is the right-continuous augmentation of the natural filtration generated by W, and that $\mathcal{F}_0$ includes all the Q-negligible events in $\mathcal{F}$. All processes in this paper are defined with reference to $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$.

$O = (O_t)$ is a non-decreasing process with the interpretation that O_t is the cumulative autonomous outflow of pollution from the firm's private stock until time t. $L = (L_t)$ is a non-decreasing process with the interpretation that L_t is the cumulative controlled emission of pollution from the firm's private stock until time t. The cumulative inflow into the firm's private stock until time t is $O_t + X_t + R_t$. $X = (X_t)$ is the Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$ with mean x, drift μ and variance σ^2, i.e., $X_t = x + \mu t + \sigma W_t$. Thus, the firm's private pollution stock process is $Z = (Z_t)$, where $Z_t = X_t + R_t - L_t$ and $R = (R_t)$ is given by

$$R_t = \sup\{[L_u - X_u]^+ \mid u \in [0, t]\} \tag{2.1.1}$$

Although R depends on L, our notation will not explicitly express this dependence. We assume the following regarding the parameters introduced so far.

Assumption 2.1.2. *Henceforth,* $x \geq 0$, $h > 0$, $l > 0$, $\alpha > 0$ *and* μ, $\sigma \in \Re$ *with* $\sigma \neq 0$.

Next, we define a feasible control process L.

Definition 2.1.3. *L is a feasible control process if it is a real-valued process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$, with sample paths that are non-negative, non-decreasing and right-continuous, and $\Delta L_t = L_t - L_{t-} \leq Z_{t-}$ for every $t \in \Re_+$. Let $\mathcal{L}(x, \mu, \sigma)$ be the set of feasible control processes.*

It immediately follows from (2.1.1) and Definition 2.1.3 that R is a real-valued process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$, with non-negative, non-decreasing and right-continuous sample paths. Moreover, (2.1.1) ensures that Z is a nonnegative process. Let the instantaneous holding cost of the pollution stock $x \in \Re_+$ be $H(x) = hx$. The penalty for emitting a unit of pollution is l.

Let $T_0 = 0$. Given $n \in \mathcal{Z}_{++}$ and a stopping time T_{n-1}, define $T_n = \inf\{t > T_{n-1} \mid Z_t \neq Z_{t-}\}$. Thus, T_n is the random time of the n-th jump in the value of Z. Associate with T_n the random variable $\Delta Z_{T_n} = Z_{T_n} - Z_{T_n-} \in \mathcal{F}_{T_n}$, which describes the size of the jump in the value of Z at T_n. Define $\Delta R^L_{T_n} = \Delta Z_{T_n} \vee 0$ and $\Delta L_{T_n} = -(\Delta Z_{T_n} \wedge 0)$. ΔR_{T_n} (resp. ΔL_{T_n}) is the size of the upward (resp. downward) jump of Z at T_n. Given $t \in \Re_+$, let $N(t) = \sup\{n \in \mathcal{Z}_+ \mid T_n \leq t\}$; this random variable counts the number of jumps of Z upto time t. Thus, $\sum_{n=0}^{N(t)} \Delta R_{T_n}$ (resp. $\sum_{n=0}^{N(t)} \Delta L_{T_n}$) is the sum of the upwards (resp. downwards) jumps in Z and R (resp. L) upto time t. These definitions allow us to decompose R and L into continuous and jump parts as follows:

$$R_t = \rho_t + \sum_{n=0}^{N(t)} \Delta R_{T_n} \qquad \text{and} \qquad L_t = \lambda_t + \sum_{n=0}^{N(t)} \Delta L_{T_n}$$

where ρ_t and λ_t are the continuous components of R_t and L_t respectively.

We note the following facts. As R and L are non-decreasing, they are of bounded variation; therefore, so is $R - L$. As W is a continuous martingale and $R - L$ is of bounded variation, Z is a semimartingale (Elliott 1982, Chapter 12). Given that all processes in this paper are adapted to $(\mathcal{F}_t)$ and right-continuous, they are progressively measurable (Elliott 1982, Theorem 2.32).

2.2 Cost formulation

We calculate the firm's cost of managing the private pollution stock as follows: a unit of pollution is charged h for every instant that it stays in the stock and is charged l when it leaves the stock. Consequently, the cost of implementing a feasible control process L until time t is

$$C_t(L; x, \mu, \sigma, h, 0, l, \alpha) = h \int_{[0,t]} ds\, e^{-\alpha s}(X_s + R_s - L_s) + l \int_{[0,t]} e^{-\alpha s} dL_s \tag{2.2.1}$$

and the cost of implementing it over $\Re_+$ is

$$C_\infty(L; x, \mu, \sigma, h, 0, l, \alpha) = \limsup_{t \uparrow \infty} C_t(L; x, \mu, \sigma, h, 0, l, \alpha)$$

Definition 2.2.2. *$L \in \mathcal{L}(x,\mu,\sigma)$ is said to be optimal given $(h,0,l,\alpha)$ if, for every $L' \in \mathcal{L}(x,\mu,\sigma)$, we have $EC_\infty(L;x,\mu,\sigma,h,0,l,\alpha) \le EC_\infty(L';x,\mu,\sigma,h,0,l,\alpha)$. If $L \in \mathcal{L}(x,\mu,\sigma)$ is optimal given $(h,0,l,\alpha)$, then let $C(x,\mu,\sigma,h,0,l,\alpha) = EC_\infty(L;x,\mu,\sigma,h,0,l,\alpha)$.*

Now consider the following cost formulation. Interpret R as the process of cumulative injections into the private stock and L as the process of cumulative emissions from the private stock. Suppose every unit of pollution injected into the stock is charged c_2 as an entry fee when it enters the stock, is charged holding cost c_1 while it stays in the stock, and is charged an emission penalty c_3 when it leaves the stock. The resulting 'cost' of implementing $L \in \mathcal{L}(x,\mu,\sigma)$ until time t is

$$\begin{aligned} C_t(L;x,\mu,\sigma,c_1,c_2,c_3,\alpha) = c_1 &\int_{[0,t]} ds\, e^{-\alpha s}(X_s + R_s - L_s) \\ &+ c_2 \int_{[0,t]} e^{-\alpha s} dR_s + c_3 \int_{[0,t]} e^{-\alpha s} dL_s \end{aligned} \tag{2.2.3}$$

The actual cost formula (2.2.1) amounts to setting $c_1 = h$, $c_2 = 0$ and $c_3 = l$.

Given the cost parameters used in (2.2.1) and the formulation (2.2.3), consider the following artificial method of calculating the 'cost' of managing the private pollution stock. Suppose every unit of pollution injected into the stock is charged the infinite horizon holding cost h/α as entry fee when it enters the stock, is charged the modified holding cost 0 while it stays in the stock, and is charged a modified emission penalty, l minus the implicit saving of the infinite horizon holding cost h/α, when it leaves the stock.[6] The resulting 'cost' of implementing $L \in \mathcal{L}(x,\mu,\sigma)$ until time t is

$$C_t(L;x,\mu,\sigma,0,h/\alpha,l-h/\alpha,\alpha) = \frac{h}{\alpha}\int_{[0,t]} e^{-\alpha s} dR_s + \left(l - \frac{h}{\alpha}\right)\int_{[0,t]} e^{-\alpha s} dL_s \tag{2.2.4}$$

which amounts to setting $c_1 = 0$, $c_2 = h/\alpha$ and $c_3 = l - h/\alpha$ in the formulation (2.2.3). The following result connects the actual cost (2.2.1) with the artificial formulation (2.2.4) which we shall use extensively in Section 2.3.

[6] This amounts to buying a consol when a particle is injected into the private stock whose instantaneous coupon exactly pays the instantaneous holding cost. When that particle exits the stock, the consol is sold, with the proceeds being used to offset a part of the emission penalty.

Lemma 2.2.5. *(A) If $L \in \mathcal{L}(x,\mu,\sigma)$, then*

$$EC_\infty(L;x,\mu,\sigma,h,0,l,\alpha) - \frac{hx}{\alpha} - \frac{h\mu}{\alpha^2}$$
$$= EC_\infty(L;x,\mu,\sigma,0,h/\alpha,l-h/\alpha,\alpha) - \frac{h}{\alpha}\limsup_{t\uparrow\infty} e^{-\alpha t}E(R-L)_t$$

(B) If $L \in \mathcal{L}(x,\mu,\sigma)$ and $S \geq 0$ are such that $Z_t = X_t + R_t - L_t \in [0,S]$ for every $t \in \Re_+$, then $\limsup_{t\uparrow\infty} e^{-\alpha t}E(R-L)_t = 0$.

(C) If $L = 0$, then $\limsup_{t\uparrow\infty} e^{-\alpha t}E(R-L)_t = \limsup_{t\uparrow\infty} e^{-\alpha t}ER_t = 0$.

This means that the actual and artificial cost formulations, (2.2.1) and (2.2.4), are identical in the limit, *modulo* a constant.

2.3 Optimal policy and the cost function

The following basic lemma is a consequence of the change-of-variable formula of stochastic calculus.

Lemma 2.3.1. *Let $\Gamma = (\sigma^2/2)D^2 + \mu D - \alpha I$. If*

$$\Phi \in \mathcal{C}^2(\Re_+,\Re) \quad \text{and} \quad E\int_{(0,t]} ds\, e^{-2\alpha s}[D\Phi(Z_s)]^2 < \infty$$

then

$$Ee^{-\alpha t}\Phi(Z_t) = \Phi(x) + E\left[\int_{(0,t]} e^{-\alpha s}D\Phi(Z_s)d(\rho-\lambda)_s + \int_{(0,t]} ds\, e^{-\alpha s}\Gamma\Phi(Z_s) + \sum_{n=0}^{N(t)} e^{-\alpha T_n}\Delta\Phi(Z)_{T_n}\right]$$

We use this lemma to characterize a lower bound on the cost of implementing a feasible control policy. We shall go on to construct a control policy whose cost attains this lower bound, implying that it is an optimal policy. Theorem 2.3.21 states the optimal control policy and the resulting cost function.

Lemma 2.3.2. *(Optimality criterion) Suppose $\Phi : \Re_+ \to \Re$ is such that*

(a) $\Phi \in \mathcal{C}^2(\Re_+,\Re)$,

(b) $-\delta \leq D\Phi \leq l - \delta$, and

(c) $\Gamma\Phi \geq 0$.

If L is a feasible control policy, then $\Phi(x) \leq EC_\infty(L; x, \mu, \sigma, 0, \delta, l-\delta, \alpha)$ for every $x \in \Re_+$.

Given $S > 0$, let $f(.; S) : \Re_+ \to \Re$ be such that

$$Df(0; S) = -\delta \qquad Df(S; S) = l - \delta \qquad \text{and} \qquad \Gamma f(x; S) = 0 \tag{2.3.3}$$

for every $x \in (0, S)$. Given $S \in \Re_{++}$, and $f(.; S)$ that solves (2.3.3), define $F(.; S) : \Re_+ \to \Re$ by

$$F(x; S) = \begin{cases} f(x; S), & \text{if } x \in [0, S] \\ f(S; S) + (l - \delta)(x - S), & \text{if } x > S \end{cases} \tag{2.3.4}$$

The roots of the characteristic polynomial of $\Gamma f(.; S)$ are $\lambda_1 = -\beta - \gamma < 0$ and $\lambda_2 = -\beta + \gamma > 0$, where

$$\beta = \mu/\sigma^2 \qquad \text{and} \qquad \gamma = (\beta^2 + 2\alpha/\sigma^2)^{1/2}$$

Given $S > 0$, the unique solution of (2.3.3) is

$$f(x; S) = ae^{\lambda_1 x} + be^{\lambda_2 x} \tag{2.3.5}$$

where

$$a = \frac{\delta e^{\gamma S} + (l - \delta)e^{\beta S}}{(\beta + \gamma)(e^{\gamma S} - e^{-\gamma S})} \qquad \text{and} \qquad b = \frac{\delta e^{-\gamma S} + (l - \delta)e^{\beta S}}{(\gamma - \beta)(e^{\gamma S} - e^{-\gamma S})} \tag{2.3.6}$$

It is clear from (2.3.3) and (2.3.4) that $F(.; S)$ and $DF(.; S)$ are continuous. Clearly,

$$D^2F(x; S) = \begin{cases} D^2 f(x; S), & \text{if } x \in (0, S) \\ 0, & \text{if } x \in (S, \infty) \end{cases}$$

Clearly, $D^2F(.; S)$ is continuous at S if and only if $\lim_{x \uparrow S} D^2F(x; S) = 0$. Elementary calculations reveal that this condition is satisfied if and only if $S > 0$ solves

$$(\delta - l)e^{\beta S}\left[\gamma \cosh(\gamma S) - \beta \sinh(\gamma S)\right] = \gamma\delta \tag{2.3.7}$$

Lemma 2.3.8. *There exists $S > 0$ that solves (2.3.7) if and only if $\delta > l$. If $S > 0$ solves (2.3.7), then it is unique; S is a function of μ, σ, h, l and α, but is independent of x.*

The next result shows that F defined by (2.3.4) satisfies assumptions (a), (b) and (c) of Lemma 2.3.2.

Lemma 2.3.9. *Suppose $\delta > l$, S is the unique solution of (2.3.7), $f(.;S)$ is defined by (2.3.5), and $F(.;S)$ is defined by (2.3.4). Then,*

(A) $F(.;S) \in \mathcal{C}^2(\Re_+, \Re)$,

(B) $-\delta \le DF(.;S) \le l - \delta$, and

(C) $\Gamma F(.;S) \ge 0$.

Combining Lemmas 2.3.2 and 2.3.9, we immediately have

Lemma 2.3.10. *Suppose $\delta > l$, S is the unique solution of (2.3.7), $f(.;S)$ is defined by (2.3.5), and $F(.;S)$ is defined by (2.3.4). Then,*

$$F(x;S) \le EC_\infty(L; x, \mu, \sigma, 0, \delta, l - \delta, \alpha) \qquad (2.3.11)$$

for every $x \in \Re_+$ and feasible control policy L.

We now construct a feasible control policy L such that equality holds in (2.3.11). By Lemma 2.3.10, this policy will be an optimal control policy.

Let $S > 0$ be given by (2.3.7). For every $t \in \Re_+$, let

$$L_t = \sup\left\{[X_u + R_u - S]^+ \mid u \in [0,t]\right\} \qquad (2.3.12)$$

This amounts to imposing an upper reflecting barrier on Z at S. Informally, L grows only at random times when Z hits the upper barrier S and X is rising, with the rise in L being just sufficient to exactly offset the growth of X.

Lemma 2.3.13. *Suppose $S > 0$.*

(A) If R and L satisfy (2.1.1) and (2.3.12) for every $t \in \Re_+$, then L is a continuous feasible control policy with $R_0 = 0$ and $L_0 = [X_0 - S]^+$.

(B) There exists a unique solution of (2.1.1) and (2.3.12).

(C) Suppose $\delta > l$, $S > 0$ is the unique solution of (2.3.7), f is defined by (2.3.3), F is defined by (2.3.4), and (R, L) is the unique solution of (2.1.1) and (2.3.12) given S. Then, for every $x \in \Re_+$,

$$EC_\infty(L; x, \mu, \sigma, 0, \delta, l - \delta, \alpha) = F(x)$$

Lemmas 2.3.10 and 2.3.13(C) yield the firm's cost function when $\delta > l$. We now turn to the problem when $\delta \le l$.

Let $\phi : \Re \to \Re$ be such that

$$D\phi(0) = -\delta \qquad \lim_{x\uparrow\infty} \phi(x) = 0 \qquad \lim_{x\uparrow\infty} D\phi(x) = 0 \qquad \text{and} \qquad \Gamma\phi(x) = 0 \tag{2.3.14}$$

for every $x \in \Re_+$. It is easy to check that the unique solution of (2.3.14) is

$$\phi(x) = \frac{\delta}{\beta + \gamma} e^{-(\beta+\gamma)x} \tag{2.3.15}$$

The following lemma notes that ϕ satisfies the assumptions of Lemma 2.3.2.

Lemma 2.3.16. *Suppose $\delta \leq l$ and ϕ is defined by (2.3.15). Then,*

(A) $\phi \in \mathcal{C}^2(\Re_+, \Re)$,

(B) $-\delta \leq D\phi \leq l - \delta$, *and*

(C) $\Gamma\phi \geq 0$.

Combining Lemmas 2.3.2 and 2.3.16 yields

Lemma 2.3.17. *Suppose $\delta \leq l$ and ϕ is defined by (2.3.15). Then,*

$$\phi(x) \leq E_0 C_\infty(L; x, \mu, \sigma, 0, \delta, l - \delta, \alpha) \tag{2.3.18}$$

for every $x \in \Re_+$ and feasible control policy L.

We now construct a feasible policy L such that equality holds in (2.3.18). This policy will be the optimal policy when $\delta \leq l$. Let

$$L_t = 0, \qquad t \in \Re_+ \tag{2.3.19}$$

This amounts to imposing no upper barrier on the internal pollution stock. The proof of the following result mimics that of Lemma 2.3.13.

Lemma 2.3.20. *(A) If (R, L) satisfies (2.1.1) and (2.3.19) for every $t \in \Re_+$, then L is a continuous feasible control policy with $(R_0, L_0) = (0, 0)$.*

(B) There exists a unique solution of (2.1.1) and (2.3.19).

(C) Suppose $\delta \leq l$, ϕ is defined by (2.3.16) and (R, L) is the unique solution of (2.1.1) and (2.3.19). Then, for every $x \in \Re_+$,

$$EC_\infty(L; x, \mu, \sigma, 0, \delta, l - \delta, \alpha) = \phi(x)$$

Lemmas 2.3.17 and 2.3.20(C) yield the firm's cost function when $\delta \leq l$. Combining Lemmas 2.2.5, 2.3.13 and 2.3.20, we have

Theorem 2.3.21. *Given parameters $(x, \mu, \sigma, h, 0, l, \alpha)$, the unique optimal control policy when $l < h/\alpha$ is given by (2.3.12) and the unique optimal control policy when $l \geq h/\alpha$ is given by (2.3.19), with cost function*

$$C(x, \mu, \sigma, h, 0, l, \alpha) = \frac{hx}{\alpha} + \frac{h\mu}{\alpha^2} + \begin{cases} F(x), & \text{if } h > l\alpha \\ \phi(x), & \text{if } h \leq l\alpha \end{cases} \tag{2.3.22}$$

We conclude this section by noting some properties of the cost function.

Theorem 2.3.23. *If $h/\alpha > l$ and $S(h, l)$ is defined by (2.3.7), then*

(A) $D_1 S(h, l) < 0$ and $D_2 S(h, l) > 0$.

Suppose C is defined by (2.3.22) and c by (1.2.1). Then,

(B) C is decreasing in α and increasing in h and l.

(C) C is concave in (h, l).

(D) $D_{12}c \geq 0$. More precisely,

$$D_{12}c(x, h) \begin{cases} > 0, & \text{if } (x, h) \in (0, S(h)) \times (l\alpha, \infty) \\ = 0, & \text{if } (x, h) \in (S(h), \infty) \times (l\alpha, \infty) \\ > 0, & \text{if } (x, h) \in (0, \infty) \times (0, l\alpha) \end{cases}$$

(E) $D_{122}c \leq 0$. More precisely,

$$D_{122}c(x, h) \begin{cases} < 0, & \text{if } (x, h) \in (0, S(h)) \times (l\alpha, \infty) \\ = 0, & \text{if } (x, h) \in (S(h), \infty) \times (l\alpha, \infty) \\ = 0, & \text{if } (x, h) \in (0, \infty) \times (0, l\alpha) \end{cases}$$

(F) Sign of $D_{112}c(x, h)$ varies. More precisely,

$$D_{112}c(x, h) \begin{cases} > 0 \vee < 0, & \text{if } (x, h) \in (0, S(h)) \times (l\alpha, \infty) \\ = 0, & \text{if } (x, h) \in (S(h), \infty) \times (l\alpha, \infty) \\ > 0, & \text{if } (x, h) \in (0, \infty) \times (0, l\alpha) \end{cases}$$

3. Regulation problem

3.1 Formulation

We employ the notation and formalism outlined in Section 1.2. The firm's value prior to environmental regulation is given by $v : \Re_+ \to \Re$, the cost of managing the firm's internal pollution stock by $c : \Re_+^2 \to \Re$, and the price paid by the firm for processing technology by $\psi : \Re_+ \to \Re$.

Assumption 3.1.2. *V, v, ψ and u satisfy the following hypotheses.*

(a) V is continuous,

(b) v is continuously differentiable on $\Re_{++}$ with $Dv - D_1 c > 0$,

(c) ψ is twice continuously differentiable on $\Re_{++}$ with $D\psi < 0$, $D^2\psi > 0$ and $D_{22}c + D^2\psi > 0$, and

(d) $\sup_{h\in\Re_+} u(x,h,0) > 0$ for every $x \in \Re_+$.

3.2 Preliminaries

The firm's utility function $u : \Re_+ \times H \times \Re \to \Re$ is given by (1.2.2) and the regulator's welfare function $W : \Re_+^2 \times \Re \to \Re$ is given by (1.2.4). The IR and IC constraints that induce participation and self-selection by all types are given by (1.2.3). If truth-telling is incentive compatible, then the regulator's welfare can be written as

$$\begin{aligned} W(x, h(x), T(x)) &= V(x) + u(x, h(x), 0) - \lambda T(x) \\ &= V(x) + (1+\lambda)u(x, h(x), 0) - \lambda U(x, x) \end{aligned} \tag{3.2.1}$$

The following is a characterization of contracts that satisfy the IC constraint for all types.

Lemma 3.2.2. *Given contract (h, T), $U(x,x) = \sup\{U(x,x') \mid x' \in \Re_+\}$ for every $x \in \Re_+$ iff. $D_2U(x,x) = 0$ for every $x \in \Re_+$ and h is non-increasing.*

Pigovian regulation

Suppose the firm pays the social cost l for each emitted unit of pollution without a transfer. Consequently, firm x selects $h \in \Re_+$ to maximize $u(x,h,0)$ subject to the constraint $u(x,h,0) \geq 0$. Assuming the optimal choice $h^*(x) > 0$, we have

$$D_2c(x, h^*(x)) + D\psi(h^*(x)) = 0 \tag{3.2.3}$$

Assumption 3.1.2(d) implies the second order condition. By Assumption 3.1.2(b), Firm x's utility is $u(x, h^*(x), 0) > 0$ and the regulator's welfare is $W(x, h^*(x), 0) = V(x) + u(x, h^*(x), 0)$. It follows from (3.2.3), Assumption 3.1.2 and Theorem 2.3.23(D) that

$$Dh^*(x) = \frac{-D_{12}c(x, h^*(x))}{D_{22}c(x, h^*(x)) + D^2\psi(h^*(x))} \leq 0$$

Complete information regulation

Alternatively, suppose the regulator has complete information. Knowing x, the regulator offers a contract $(h, T) = (h_x, T_x)$, which is chosen to maximize $W(x, h, T)$ subject to the IR constraint $u(x, h, T) \geq 0$. (h_x, T_x) is characterized by the conditions

$$T_x = \psi(h_x) + c(x, h_x) - v(x) \qquad \text{and} \qquad D_2 c(x, h_x) + D\psi(h_x) = 0 \tag{3.2.4}$$

The second condition is identical to (3.2.3), which implies that $h_x = h^*(x)$. The first condition amounts to setting $u(x, h_x, T_x) = u(x, h_x, 0) + T_x = 0$, i.e., $T_x = -u(x, h_x, 0) = -u(x, h^*(x), 0) < 0$. The regulator's welfare is

$$W(x, h_x, T_x) = V(x) + (1 + \lambda)u(x, h_x, 0) = V(x) + (1 + \lambda)u(x, h^*(x), 0) > W(x, h^*(x), 0)$$

Consequently, social welfare is higher under complete information regulation than under Pigovian regulation.

3.3 Contracting under incomplete information: discrete case

Assumption 3.3.1. *$\underline{x}$, $\bar{x} \in \Re_+$ such that $\underline{x} < \bar{x}$, and F is a distribution function on $\Re_+$ such that* $\operatorname{supp} F = \{\underline{x}, \bar{x}\}$ *and $F(\underline{x}) = p \in (0, 1)$.*

In this section, the firm's type is $x \in \{\underline{x}, \bar{x}\}$, which is private information, and F is the regulator's belief about x, which is common knowledge. Consider an equilibrium in which the regulator offers a contract $\{(\underline{h}, \underline{T}), (\bar{h}, \bar{T})\}$, firm $\underline{x}$ chooses $(\underline{h}, \underline{T})$ and firm $\bar{x}$ chooses $(\bar{h}, \bar{T})$. In such an equilibrium, the following conditions must hold:

$$u(\underline{x}, \underline{h}, \underline{T}) \geq 0 \tag{3.3.2}$$

$$u(\bar{x}, \bar{h}, \bar{T}) \geq 0 \tag{3.3.3}$$

$$u(\underline{x}, \underline{h}, \underline{T}) \geq u(\underline{x}, \bar{h}, \bar{T}) \tag{3.3.4}$$

$$u(\bar{x}, \bar{h}, \bar{T}) \geq u(\bar{x}, \underline{h}, \underline{T}) \tag{3.3.5}$$

(3.3.2) and (3.3.3) are the IR constraints for the two types, and (3.3.4) and (3.3.5) are their IC constraints. Define $\Phi : \Re_+ \to \Re$ by

$$\Phi(h) = u(\bar{x}, h, 0) - u(\underline{x}, h, 0) = \int_{[\underline{x}, \bar{x}]} dx \, [Dv(x) - D_1 c(x, h)]$$

It follows from Theorem 2.3.23 that $D\Phi(h) = -\int_{[\underline{x},\bar{x}]} dx\, D_{12}c(x,h) \leq 0$ and $D^2\Phi(h) = -\int_{[\underline{x},\bar{x}]} dx\, D_{122}c(x,h) \geq 0$. Given $\underline{U} = u(\underline{x},\underline{h},\underline{T})$ and $\bar{U} = u(\bar{x},\bar{h},\bar{T})$, constraints (3.3.2) to (3.3.5) can be re-written as

$$\underline{U} \geq 0 \tag{3.3.6}$$

$$\bar{U} \geq 0 \tag{3.3.7}$$

$$\underline{U} \geq \bar{U} - \Phi(\bar{h}) \tag{3.3.8}$$

$$\bar{U} \geq \underline{U} + \Phi(\underline{h}) \tag{3.3.9}$$

The regulator's optimal contract $\{(\underline{U},\underline{h}),(\bar{U},\bar{h})\}$ maximizes

$$p\,[(1+\lambda)u(\underline{x},\underline{h},0) - \lambda\underline{U}] + (1-p)\left[(1+\lambda)u(\bar{x},\bar{h},0) - \lambda\bar{U}\right] \tag{3.3.10}$$

subject to constraints (3.3.6) to (3.3.9).

(3.3.6) and (3.3.9) imply $\bar{U} \geq \underline{U} + \Phi(\underline{h}) \geq \Phi(\underline{h}) \geq 0$. Thus, (3.3.7) is satisfied if (3.3.6) and (3.3.9) are satisfied. If (3.3.7) is binding, then $0 \geq \underline{U} + \Phi(\underline{h})$, i.e., $\underline{U} \leq -\Phi(\underline{h}) < 0$, which violates (3.3.6). Thus, $\bar{U} > 0$. If $\underline{U} > 0$ at the optimum, then both $\underline{U}$ and $\bar{U}$ can be reduced by $\epsilon > 0$, sufficiently small, without violating constraints (3.3.6) to (3.3.9). As this increases the value of the objective, we have a contradiction. Consequently, at the optimum, we must have $\underline{U} = 0$ and $\Phi(\underline{h}) \leq \bar{U} \leq \Phi(\bar{h})$. Clearly, at the optimum, we must have $\bar{U} = \Phi(\underline{h})$. This simplifies the regulator's problem to: choose $\underline{h}$ and $\bar{h}$ to maximize

$$p(1+\lambda)u(\underline{x},\underline{h},0) + (1-p)\left[(1+\lambda)u(\bar{x},\bar{h},0) - \lambda\Phi(\underline{h})\right]$$

The first-order conditions characterizing the optimal choices are

$$D_2c(\bar{x},\bar{h}) + D\psi(\bar{h}) = 0 \tag{3.3.11}$$

and

$$D_2c(\underline{x},\underline{h}) + D\psi(\underline{h}) = -\frac{1-p}{p}\frac{\lambda}{1+\lambda}D\Phi(\underline{h}) \tag{3.3.12}$$

As $D_{12}c \geq 0$, we have $D\Phi \leq 0$ and

$$D_2c(\bar{x},\underline{h}) + D\psi(\underline{h}) \geq D_2c(\underline{x},\underline{h}) + D\psi(\underline{h}) \geq 0 = D_2c(\bar{x},\bar{h}) + D\psi(\bar{h})$$

Since $D_{22}c(\bar{x},.) + D^2\psi(.) > 0$, we have $\underline{h} \geq \bar{h}$. Comparing (3.3.11) and (3.3.12) with (3.2.3), we have $\bar{h} = h_{\bar{x}}$ and $\underline{h} \geq h_{\underline{x}}$, i.e., relative to the full information choices, type $\bar{x}$'s choice is not distorted while type $\underline{x}$'s choice is distorted upwards. We collect these facts in the following result.

Theorem 3.3.13. *Given Assumptions 3.1.2 and 3.3.1, $\{(\underline{U}, \underline{h}), (\bar{U}, \bar{h})\}$ is the optimal contract, where $\bar{h}$ and $\underline{h}$ are characterized by (3.3.11) and (3.3.12) respectively, $\bar{U} = \Phi(\underline{h})$ and $\underline{U} = 0$. The implied transfers are $\bar{T} = \Phi(\underline{h}) + c(\bar{x}, \bar{h}) + \psi(\bar{h}) - v(\bar{x})$ and $\underline{T} = c(\underline{x}, \underline{h}) + \psi(\underline{h}) - v(\underline{x})$. Moreover, $\bar{h} = h_{\bar{x}} \leq h_{\underline{x}} \leq \underline{h}$.*

3.4 Contracting under incomplete information: continuum case

In this section we consider the optimal mechanism design problem with a continuum of types. Our first objective is to derive necessary conditions that an optimal mechanism must satisfy, assuming that one exists. Our analysis of these conditions yields useful qualitative information about the optimal mechanism, which is reported in Theorem 3.4.14. Our second objective is Theorem 3.4.15 which establishes the existence of an optimal mechanism.

Assumption 3.4.1. *F is a distribution function on $\Re_+$ such that*

(a) $\operatorname{supp} F = X = [x_0, x_1] \subset \Re_{++}$, *and*

(b) F is twice differentiable, with $f(x) = DF(x) > 0$ and $DG(x) < 0$ for $x \in (x_0, x_1)$, where $G(x) = [1 - F(x)]/f(x)$.

F is the regulator's belief about the firm's type x. (a) serves two main purposes. First, it is used in the proof of Theorem 3.4.15. Secondly, it is a simplifying assumption that permits sharper statement of results; for instance, if X is replaced by $\Re_+$, then some of the strict inequalities have to be weakened and some statements in this section (e.g., transversality conditions) have to be replaced by analogous limit statements. $DG < 0$ is a standard monotone hazard rate condition that is satisfied by many familiar distributions (Bagnoli and Bergstrom 1989). The following assumption is used in the proof of Theorem 3.4.15.

Assumption 3.4.2. $h \in [0, \bar{h}]$.

Consider a Bayesian equilibrium in which the regulator offers a contract (h, T) and firm $x \in X$ chooses to participate and self-selects by choosing $(h(x), T(x))$; the restriction to the truth-telling equilibrium is without loss of generality because of the revelation principle. By Lemma 3.2.2, (h, T) makes truth-telling incentive compatible if and only if, for every $x \in X$,

$$D_2U(x, x) = 0 \qquad Dh(x) = -y(x) \qquad y(x) \geq 0 \tag{3.4.3}$$

Setting $\mathcal{U}(x) = U(x,x)$, (3.4.3) is equivalent to: for every $x \in X$,

$$D\mathcal{U}(x) = Dv(x) - D_1 c(x, h(x)) \qquad Dh(x) = -y(x) \qquad y(x) \geq 0 \qquad (3.4.4)$$

We postulate that (h,T) induces participation by every $x \in X$ if and only if

$$\mathcal{U}(x) \geq 0 \qquad (3.4.5)$$

Since $\mathcal{U}(x) = U(x,x) = v(x) - c(x,h(x)) - \psi \circ h(x) + T(x)$, we may specify a contract as $(h,\mathcal{U})$ instead of (h,T). In the formal specification of the regulator's problem, we shall treat y as the control variable and $(h,\mathcal{U})$ as the state variable.

Definition 3.4.6. *$(h,\mathcal{U};y)$ is an admissible state-control pair if it satisfies (3.4.4), (3.4.5) and the following conditions:*

(a) $(h,\mathcal{U})$ is absolutely continuous,[7]

(b) y is measurable, and

(c) $h(x) \in [0,\bar{h}]$ for every $x \in X$.

Using (3.2.1), the regulator's problem can be formulated as: find an admissible state-control pair $(h,\mathcal{U};y)$ such that $I[h,\mathcal{U};y] \leq I[\hat{h},\hat{\mathcal{U}};\hat{y}]$ for every admissible state-control pair $(\hat{h},\hat{\mathcal{U}};\hat{y})$, where

$$I[\hat{h},\hat{\mathcal{U}};\hat{y}] = -\int_X dx\, f(x) \left[V(x) + (1+\lambda)\left(v(x) - c(x,\hat{h}(x)) - \psi \circ \hat{h}(x)\right) - \lambda\hat{\mathcal{U}}(x)\right]$$

Assumption 3.1.2(c) implies that $D\mathcal{U}(x) > 0$. Consequently,

$$\mathcal{U}(x_0) = 0 \qquad (3.4.7)$$

implies (3.4.5). Conversely, suppose $(h,\mathcal{U};y)$ is an admissible state-control pair that solves the regulator's problem and $\mathcal{U}(x_0) > 0$. Define $\mathcal{U}'$ by $\mathcal{U}'(x) = \mathcal{U}(x) - \mathcal{U}(x_0)$. Then, $(h,\mathcal{U}';y)$ is an admissible state-control pair and $I[h,\mathcal{U}';y] = I[h,\mathcal{U};y] - \lambda\mathcal{U}(x_0) < I[h,\mathcal{U};y]$, a contradiction. So, if $(h,\mathcal{U};y)$ is an admissible state-control pair that solves the regulator's problem, then $\mathcal{U}$ satisfies (3.4.7).

---

[7] See Section 2.1 in Cesari 1983 for a definition. If a function is absolutely continuous, then it is continuous, differentiable Lebesgue almost everywhere on its domain, Lebesgue integrable, and the "fundamental theorem of calculus" holds for it. The class of absolutely continuous functions is the largest class of functions possessing these properties.

Remark. *In Definition 3.4.6, (3.4.5) can be replaced by (3.4.7) without loss of generality.*

Suppose $(h, \mathcal{U}; y)$ solves the regulator's problem and $h(x_0) < \bar{h}$. We derive the usual necessary conditions. The Hamiltonian function for this Lagrange problem is

$$H(x, h, \mathcal{U}; \eta, \xi; y) = -\rho f(x)\left[V(x) - (1+\lambda)\big(c(x,h) + \psi(h) - v(x)\big) - \lambda\mathcal{U}\right] + \eta[Dv(x) - D_1 c(x,h)] - \xi y$$

Applying Pontryagin's theorem (Cesari 1983, Theorem 5.1.i), there exists an absolutely continuous function $(\rho, \xi, \eta) : X \to \Re^3$ such that ρ is a constant, $\rho \geq 0$, $(\rho, \xi(x), \eta(x)) \neq 0$ for some $x \in X$,

$$D\xi(x) = -\rho f(x)(1+\lambda)\left[D_2 c(x, h(x)) + D\psi \circ h(x)\right] + \eta(x) D_{12} c(x, h(x)) \tag{3.4.8}$$

and

$$D\eta(x) = -\rho\lambda f(x) \tag{3.4.9}$$

for almost every[8] $x \in X$. Given that x_0 and x_1 are fixed, we have the transversality condition: $\xi(x_0)dh(x_0) + \eta(x_0)d\mathcal{U}(x_0) - \xi(x_1)dh(x_1) - \eta(x_1)d\mathcal{U}(x_1) = 0$ for all feasible variations $(dh(x_0), d\mathcal{U}(x_0), dh(x_1), d\mathcal{U}(x_1))$. As $\mathcal{U}(x_0)$ is fixed at 0, $d\mathcal{U}(x_0)$ is identically equal to 0. Thus, $\eta(x_0)$ is unrestricted. As $\mathcal{U}(x_1)$ is unrestricted, we have $\eta(x_1) = 0$. This simplifies the transversality condition to $\xi(x_0)dh(x_0) = \xi(x_1)dh(x_1)$ for all feasible variations $(dh(x_0), dh(x_1))$. It follows that $\xi(x_0) = \xi(x_1) = 0$.

Since $\eta(x_1) = 0$, we have

$$-\eta(x) = \eta(x_1) - \eta(x) = \int_{[x,x_1]} dy\, D\eta(y) = -\rho\lambda \int_{[x,x_1]} dx\, f(x) = -\rho\lambda[1 - F(x)] \tag{3.4.10}$$

Suppose $\rho = 0$. Then, (3.4.10) implies $\eta(x) = 0$ for every $x \in X$. Thus, (3.4.8) implies $D\xi(x) = 0$ for every $x \in X$. As $\xi(x_0) = 0$, it follows that $\xi(x) = \int_{[x_0,x]} dy\, D\xi(y) = 0$ for every $x \in X$. This means $(\rho, \xi(x), \eta(x)) = 0$ for every $x \in X$, a contradiction. Therefore, $\rho > 0$. Without loss of generality, let $\rho = 1$. Consequently, $\eta(x) = \lambda[1 - F(x)]$ for every $x \in X$, i.e., η decreases monotonically from λ at x_0 to 0 at x_1.

Moreover, for almost every $x \in X$, $y = y(x)$ minimizes $H(x, h(x), \mathcal{U}(x); \xi(x), \eta(x); y)$ subject to the constraint $y \geq 0$. This is equivalent to $y = y(x)$ maximizing $\xi(x)y$ subject

[8] In this section, "almost all/every" refers to Lebesgue measure.

to the constraint $y \geq 0$. If $\xi(x) > 0$, then a maximum does not exist. Consequently, we must have $\xi(x) \leq 0$ for almost every $x \in X$.

Regime 1: $y(x) > 0$

Consider $x \in X$ such that $y(x) > 0$. It follows that $\xi(x) = 0$. Since ξ attains a maximum at x, we have $D\xi(x) = 0$. (3.4.8) and (3.4.10) imply

$$D_2 c(x, h(x)) + D\psi \circ h(x) = \frac{\lambda}{1+\lambda} G(x) D_{12} c(x, h(x)) \tag{3.4.11}$$

(3.4.11) implicitly defines $h(x)$ and (3.4.4) determines $\mathcal{U}(x)$. It follows from (3.4.11) that

$$Dh(x) = \frac{\lambda G(x) D_{112} c(x, h(x)) + [\lambda DG(x) - (1+\lambda)] D_{12} c(x, h(x))}{(1+\lambda)[D_{22} c(x, h(x)) + D^2\psi \circ h(x)] - \lambda G(x) D_{122} c(x, h(x))} = -y(x) < 0 \tag{3.4.12}$$

Assumption 3.1.2 and Theorem 2.3.23 imply that the denominator is positive. Assumption 3.4.1 and Theorem 2.3.23 imply that the second term in the numerator is negative, while the first term can be positive or negative.

To sum up, in Regime 1, h is positive and decreasing, ξ is zero, $\mathcal{U}$ is non-negative and increasing, and η is positive and decreasing.

Regime 2: $y(x) = 0$

Consider $x \in X$ such that $y(x) = 0$. Let

$$x' = \sup(X - y^{-1}(0)) \cap [x_0, x] \qquad \text{and} \qquad x'' = \inf(X - y^{-1}(0)) \cap [x, x_1]$$

By definition, there exists a sequence (x_n) in $(X - y^{-1}(0)) \cap [x_0, x]$ such that $\lim_{n\uparrow\infty} x_n = x'$. By definition, $y(x_n) > 0$ for every $n \in \mathcal{Z}_{++}$. It follows that $\xi(x_n) = 0$ for every $n \in \mathcal{Z}_{++}$. By the continuity of ξ, $\xi(x') = \lim_{n\uparrow\infty} \xi(x_n) = 0$. Similarly, $\xi(x'') = 0$. As $Dh(x) = -y(x) = 0$ for every $x \in (x', x'')$, $h(x)$ is some constant $\kappa \geq 0$ for every $x \in (x', x'')$. (3.4.8) yields

$$\xi(x) = (1+\lambda) \int_{[x',x]} dy\, f(y) \left[D_2 c(y, \kappa) + D\psi(\kappa) - \frac{\lambda}{1+\lambda} G(y) D_{12} c(y, \kappa) \right]$$

The condition $\xi(x'') = 0$ fixes the value of κ *via* the equation

$$\int_{[x',x'']} dy\, f(y) \left[D_2 c(y, \kappa) + D\psi(\kappa) - \frac{\lambda}{1+\lambda} G(y) D_{12} c(y, \kappa) \right] = 0 \tag{3.4.13}$$

To sum up, in Regime 2, h is a constant $\kappa \geq 0$, ξ is non-positive, $\mathcal{U}$ is non-negative and increasing, and η is positive and decreasing.

Theorem 3.4.14. *Suppose Assumptions 3.1.2 and 3.4.1 hold, and $\{h, \mathcal{U}\}$ is the regulator's optimal contract. Then,*

(A) η decreases monotonically from λ at x_0 to 0 at x_1; set $\rho = 1$ in (3.4.10).

(B) ξ is non-positive, with $\xi(x_0) = 0 = \xi(x_1)$; it is positive only if $Dh(x) = 0$; set $\rho = 1$ in (3.4.8).

(C) $\mathcal{U}$ is increasing, with $\mathcal{U}(x_0) = 0$; see (3.4.4).

(D) h is non-increasing; it is determined by (3.4.11) if $Dh(x) < 0$, and by (3.4.13) if $Dh(x) = 0$.

(E) If $(x, h(x))$ is such that $h(x) > l\alpha$ and $x > S(h(x))$, then $Dh(x) = 0$.

(F) If x is such that $Dh(x) < 0$, then $h(x) > h^*(x)$.

We have so far characterized an optimal contract assuming that one exists. So, is there an optimal contract? We answer this question in two steps: (a) the set of admissible state-control pairs is nonempty, and (b) there is an admissible state-control pair that solves the regulator's problem.

Consider the state-control pair $(h, \mathcal{U}; y)$, where $h(x) = \bar{h}/2$, $\mathcal{U}(x) = \int_{[x_0, x]} dy\, [Dv(y) - D_1 c(y, \bar{h}/2)]$ and $y(x) = 0$ for every $x \in X$. It is trivial to check that $(h, \mathcal{U}; y)$ is an admissible state-control pair.

The following existence result is an application of an extension of Filippov's existence theorem (Cesari 1983, Theorem 9.3.i).

Theorem 3.4.15. *Given Assumptions 3.1.2 and 3.4.1, there exists an admissible state-control pair $(h, \mathcal{U}; y)$ such that $I[h, \mathcal{U}; y] \geq I[h', \mathcal{U}'; y']$ for every admissible state-control pair $(h', \mathcal{U}'; y')$.*

4. Extensions

Variations on the above regulatory model are possible by replacing x with μ, σ or α as the firm's private information. The choice of x as private information allows rather straightforward analytical solution of the model. Our analysis of the other choices, which we do not report in this paper, points to the resulting model being very difficult to analyze with purely analytical techniques; the reason is that S varies with μ, σ and α, but not with x. The other cases require analytical techniques to be supplemented by exact/approximate numerical techniques for the determination of the optimal contract.

An important theoretical extension of this model would be to consider a situation in which a regulator faces many firms. Even if the firms do not interact directly, they will be connected *via* the regulator's budget constraint and the fact that all the firms' emissions add to the same public stock of pollution.

Another theoretical extension is to endogenize the mandated clean-up technology for public pollution by choosing l to maximize some specified social welfare function.

Finally, numerical analysis and application of the solution proposed in this paper would be of great interest. For instance, one could numerically calculate and analyze Regimes 1 and 2 of Section 3.4. Since the firm's pollution-producing technology is represented by a few easily measurable statistical parameters, this sparse and direct, yet flexible, conceptualization of the relevant aspects of the firm should allow straightforward quantitative application of our results.

Appendix

Proof of Lemma 2.2.5. (A) (2.2.2) and the definition of X imply

$$C_t(L;x,\mu,\sigma,h,0,l,\alpha) = h\int_{[0,t]} ds\, e^{-\alpha s}(x+\mu s+\sigma W_s+R_s-L_s) + l\int_{[0,t]} ds\, e^{-\alpha s} dL_s$$

We have $\alpha\int_{[0,t]} ds\, e^{-\alpha s}R_s = \alpha\int_{[0,t]} ds\, e^{-\alpha s}\left(\rho_s+\sum_{n=0}^{N(s)}\Delta R_{T_n}\right)$. As $\rho_0=0$, we have $\alpha\int_{[0,t]} ds\, e^{-\alpha s}\rho_s = \int_{[0,t]} e^{-\alpha s}d\rho_s - e^{-\alpha t}\rho_t$ and

$$\begin{aligned}
\alpha\int_{[0,t]} ds\, e^{-\alpha s}\sum_{n=0}^{N(s)}\Delta R_{T_n} &= \alpha\int_{\Re_+} ds\, e^{-\alpha s}1_{[0,t]}(s)\sum_{n=0}^{\infty}\Delta R_{T_n}1_{[0,s]}(T_n)\\
&= \sum_{n=0}^{\infty}\Delta R_{T_n}\,\alpha\int_{\Re_+} ds\, e^{-\alpha s}1_{[0,t]}(s)1_{[T_n,\infty)}(s)\\
&= \sum_{n=0}^{\infty}\Delta R_{T_n}1_{[0,t]}(T_n)\alpha\int_{\Re_+} ds\, e^{-\alpha s}1_{[T_n,t]}(s)\\
&= \sum_{n=0}^{\infty}\Delta R_{T_n}1_{[0,t]}(T_n)\alpha\int_{[T_n,t]} ds\, e^{-\alpha s}\\
&= \sum_{n=0}^{N(t)}\Delta R_{T_n}(e^{-\alpha T_n}-e^{-\alpha t})
\end{aligned}$$

as $N(t)$ is finite almost surely. Thus,

$$\begin{aligned}
\alpha\int_{[0,t]} ds\, e^{-\alpha s}R_s &= \int_{[0,t]} e^{-\alpha s}d\rho_s - e^{-\alpha t}\rho_t + \sum_{n=0}^{N(t)}\Delta R_{T_n}(e^{-\alpha T_n}-e^{-\alpha t})\\
&= \int_{[0,t]} e^{-\alpha s}dR_s - e^{-\alpha t}R_t
\end{aligned}$$

The integral $\int_{[0,t]} ds\, e^{-\alpha s}L_s$ is manipulated analogously. Therefore,

$$\begin{aligned}
C_t(L;x,\mu,\sigma,h,0,l,\alpha) - \left(\frac{hx}{\alpha}+\frac{h\mu}{\alpha^2}\right)(1-e^{-\alpha t}) + \frac{h\mu}{\alpha}te^{-\alpha t} - h\sigma\int_{[0,t]} ds\, e^{-\alpha s}W_s&\\
= \frac{h}{\alpha}\int_{[0,t]} e^{-\alpha s}dR_s + \left(l-\frac{h}{\alpha}\right)\int_{[0,t]} ds\, e^{-\alpha s}dL_s - \frac{h}{\alpha}e^{-\alpha t}(R-L)_t&\\
= C_t(L;x,\mu,\sigma,0,h/\alpha,l-h/\alpha,\alpha) - \frac{h}{\alpha}e^{-\alpha t}(R-L)_t&
\end{aligned}$$

Taking expectations and letting $t\uparrow\infty$, yields the desired formula.

(B) Note that $E(R-L)_t = E(Z-X)_t$. Clearly, $\limsup_{t\uparrow\infty} e^{-\alpha t} EZ_t = 0$ as Z is bounded. As $EX_t = x + \mu t + EW_t = x + \mu t$, we have $\limsup_{t\uparrow\infty} e^{-\alpha t} EX_t = 0$.

(C) In this proof we shall appeal to some formulae stated in Karatzas and Shreve (1988). While the cited formulae refer to the maximum process and the process of passage times associated with the Wiener process, we adapt the formulae for our purposes without formal proof by appealing to the strong Markov property and reflection principle associated with the Wiener process (Karatzas and Shreve 1988, Section 2.6).

Let $L = 0$. Define $m_t^X = \inf\{X_u \mid u \in [0,t]\}$ for $t \geq 0$, and $\tau_0 = \inf\{t \geq 0 \mid X_t = 0\}$. As $x \geq 0$, we have $0 \leq R_t = -m_{t\vee\tau_0}^X$ for every $t \geq 0$.

First consider the problem with $\mu < 0$. Let $\tau^* = \inf\{t \geq -x/\mu \mid W_t = 0\}$. By the strong Markov property of W, we have $\tau^* < \infty$, Q-a.s. (Karatzas and Shreve 1988, Remark 8.3 in Section 2.8). As $\mu < 0$, we have $X_{\tau^*} = x + \mu\tau^* + \sigma W_{\tau^*} = x + \mu\tau^* \leq x + \mu(-x/\mu) = 0$. Therefore, $\tau_0 \leq \tau^*$. It follows that $t \vee \tau_0 \leq t \vee \tau^*$ for every $t \geq 0$, which implies $0 \leq R_t = -m_{t\vee\tau_0}^X \leq -m_{t\vee\tau^*}^X$ for every $t \geq 0$. Thus, it is sufficient to show that $\lim_{t\uparrow\infty} e^{-\alpha t} E m_{t\vee\tau^*}^X = 0$.

Let $m_t^W = \inf\{W_u \mid u \in [0,t]\}$ for $t \geq 0$. Define the passage time $\tau_n = \inf\{t \geq 0 \mid W_t = -n\}$ for $n \in \mathcal{Z}_{++}$. It is easy to confirm that $Ee^{-\alpha\tau_n} m_{\tau_n}^W = -nEe^{-\alpha\tau_n} = -ne^{-n\sqrt{2\alpha}}$ (Karatzas and Shreve 1988, (8.6) in Section 2.8). Consequently,

$$\lim_{n\uparrow\infty} Ee^{-\alpha\tau_n} m_{\tau_n}^W = 0 \tag{A.1}$$

Consider an increasing sequence $(t_k) \subset \Re_+$ such that $\lim_{k\uparrow\infty} t_k = \infty$. Note that

$$Q\left[\lim_{k\uparrow\infty} m_{t_k}^W = -\infty\right] = Q\left[\bigcap_{n\in\mathcal{Z}_{++}} \left[\lim_{k\uparrow\infty} m_{t_k}^W < -n\right]\right] = Q\left[\bigcap_{n\in\mathcal{Z}_{++}} \bigcup_{k\in\mathcal{Z}_{++}} [m_{t_k}^W < -n]\right]$$

Using the monotone convergence theorem, for every $n \in \mathcal{Z}_{++}$,

$$Q\left[\bigcup_{k\in\mathcal{Z}_{++}} [m_{t_k}^W < -n]\right] = Q\left[\lim_{K\uparrow\infty} \bigcup_{k=1}^{K} [m_{t_k}^W < -n]\right] = \lim_{K\uparrow\infty} Q\left[\bigcup_{k=1}^{K} [m_{t_k}^W < -n]\right]$$

which equals (Karatzas and Shreve 1988, (8.4) in Section 2.8)

$$\lim_{K\uparrow\infty} Q[m_{t_K}^W < -n] = \lim_{K\uparrow\infty} \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{-n/\sqrt{t_K}} dx\, e^{-x^2/2} = \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{0} dx\, e^{-x^2/2} = 1$$

It follows that $Q\left[\lim_{k\uparrow\infty} m^W_{t_k} = -\infty\right] = 1$.

For $k \in \mathcal{Z}_{++}$, let $n(k) = \sup\{n \in \mathcal{Z}_+ \mid -n \geq m^W_{t_k}\}$. As $\lim_{k\uparrow\infty} m^W_{t_k} = -\infty$, Q-a.s., we have $\lim_{k\uparrow\infty} n(k) = \infty$, Q-a.s.

If $t_k \leq \tau^*$, then $m^W_{t_k\vee\tau^*} = m^W_{\tau^*} = 0 > -(n(k)+1) = m^W_{\tau_{n(k)}} - 1$. If $t_k > \tau^*$, then $m^W_{t_k\vee\tau^*} = m^W_{t_k} > -(n(k)+1) = m^W_{\tau_{n(k)}} - 1$. Thus, for every $k \in \mathcal{Z}_{++}$,

$$m^W_{t_k\vee\tau^*} > m^W_{\tau_{n(k)}} - 1 \tag{A.2}$$

As $\tau_{n(k)} \leq t_k < \tau_{n(k)+1}$ for every $k \in \mathcal{Z}_{++}$, we have

$$e^{-\alpha t_k} \leq e^{-\alpha\tau_{n(k)}} \qquad \text{and} \qquad e^{-\alpha t_k} m^W_{\tau_{n(k)}} \geq e^{-\alpha\tau_{n(k)}} m^W_{\tau_{n(k)}} \tag{A.3}$$

for every $k \in \mathcal{Z}_{++}$. As $\mu < 0$, (A.2) implies

$$0 \geq m^X_{t_k\vee\tau^*} \geq x + \mu(t_k \vee \tau^*) + \sigma m^W_{t_k\vee\tau^*} > x + \mu(t_k\vee\tau^*) + \sigma(m^W_{\tau_{n(k)}} - 1)$$

Thus, (A.3) implies $0 \geq e^{-\alpha t_k} m^X_{t_k\vee\tau^*} \geq (x-\sigma)e^{-\alpha t_k} + \mu e^{-\alpha t_k}(t_k\vee\tau^*) + \sigma e^{-\alpha\tau_{n(k)}} m^W_{\tau_{n(k)}}$. Taking expectations and noting that $x \mapsto t_k \vee x$ is a convex function, Jensen's inequality yields

$$0 \geq e^{-\alpha t_k} E m^X_{t_k\vee\tau^*} \geq (x-\sigma)e^{-\alpha t_k} + \mu e^{-\alpha t_k}(t_k \vee E\tau^*) + \sigma E e^{-\alpha\tau_{n(k)}} m^W_{\tau_{n(k)}}$$

As $\tau^* < \infty$, Q-a.s., we have $E\tau^* < \infty$. As $\lim_{k\uparrow\infty} t_k = \infty$, the first two terms vanish. As $(\tau_{n(k)})_{k\in\mathcal{Z}_{++}}$ is a subsequence of $(\tau_n)_{n\in\mathcal{Z}_{++}}$, (A.1) implies $\lim_{k\uparrow\infty} E e^{-\alpha t_k} m^X_{t_k\vee\tau^*} = 0$, as required.

Now consider the problem with $\mu \geq 0$. For $t < \tau_0$, $R_t = -m^X_{t\vee\tau_0} = -m^X_{\tau_0} = 0 \leq -\sigma m^W_t$. As $x \geq 0$ and $\mu \geq 0$, we have $X_t = x + \mu t + \sigma W_t \geq \sigma W_t$ for every $t \geq 0$, which implies $m^X_t \geq \sigma m^W_t$ for every $t \geq 0$. Therefore, for $t \geq \tau_0$, we have $R_t = -m^X_{t\vee\tau_0} = -m^X_t \leq -\sigma m^W_t$. Therefore, $0 \leq e^{-\alpha t} E R_t \leq -\sigma e^{-\alpha t} E m^W_t$ for every $t \geq 0$. Using the reflection principle, we calculate that $E m^W_t = -(2t/\pi)^{1/2}$ (Karatzas and Shreve 1988, (8.3) in Section 2.8). It follows that $\lim_{t\uparrow\infty} e^{-\alpha t} E R_t = 0$. ■

Proof of Lemma 2.3.1. By the change of variable formula (Elliott 1982, Theorem 12.21),

$$\Phi(Z_t) = \Phi(Z_0) + \int_{(0,t]} D\Phi(Z_{s-})dZ_s + \frac{1}{2}\int_{(0,t]} D^2\Phi(Z_{s-})d\langle Z^c, Z^c\rangle_s + \sum_{n=1}^{N(t)} [\Delta\Phi(Z)_{T_n} - D\Phi(Z_{T_n-})\Delta Z_{T_n}]$$

where (Z_t^c) is the continuous martingale part of (Z_t). For $s > 0$, we have

$$Z_s = X_s + \rho_s - \lambda_s + \Delta Z_{T_0} + \sum_{n=1}^{N(s)} \Delta Z_{T_n}$$

where the last term can be re-written as

$$\sum_{n=1}^{N(s)} \Delta Z_{T_n} = \sum_{n=1}^{\infty} \Delta Z_{T_n} 1_{(0,s]}(T_n) = \sum_{n=1}^{\infty} \Delta Z_{T_n} \int_{(0,s]} \delta_{T_n}(du)$$

δ_{T_n} is the Dirac measure sitting at T_n. Analogously,

$$\sum_{n=1}^{N(s)} \Delta\Phi(Z)_{T_n} = \sum_{n=1}^{\infty} \Delta\Phi(Z)_{T_n} \int_{(0,s]} \delta_{T_n}(du) \qquad (A.4)$$

Therefore, for $s > 0$, $dZ_s = dX_s + d(\rho - \lambda)_s + \sum_{n=1}^{\infty} \Delta Z_{T_n} \delta_{T_n}(ds)$. As $Z_s^c = x + \sigma W_s$, we have $d\langle Z^c, Z^c \rangle_s = d\langle \sigma W, \sigma W \rangle_s = \sigma^2 d\langle W, W \rangle_s = \sigma^2 ds$. Therefore,

$$\Phi(Z_t) = \Phi(Z_0) + \int_{(0,t]} D\Phi(Z_{s-}) \left[\mu ds + \sigma dW_s + d(\rho - \lambda)_s + \sum_{n=1}^{\infty} \Delta Z_{T_n} \delta_{T_n}(ds) \right]$$
$$+ \frac{\sigma^2}{2} \int_{(0,t]} ds\, D^2\Phi(Z_{s-}) + \sum_{n=1}^{N(t)} [\Delta\Phi(Z)_{T_n} - D\Phi(Z_{T_n-}) \Delta Z_{T_n}]$$

Note that

$$\int_{(0,t]} \left(\sum_{n=1}^{\infty} \Delta Z_{T_n} \delta_{T_n} \right) (ds) D\Phi(Z_{s-}) = \sum_{n=1}^{\infty} \Delta Z_{T_n} \int_{(0,t]} \delta_{T_n}(ds) D\Phi(Z_{s-})$$
$$= \sum_{n=1}^{\infty} \Delta Z_{T_n} D\Phi(Z_{T_n-}) 1_{(0,t]}(T_n)$$
$$= \sum_{n=1}^{N(t)} \Delta Z_{T_n} D\Phi(Z_{T_n-})$$

Using this formula, cancelling terms, and using the fact that the continuity of integrators allows us to replace Z_{s-} by Z_s, we have

$$\Phi(Z_t) = \Phi(Z_0) + \sigma \int_{(0,t]} D\Phi(Z_s) dW_s + \int_{(0,t]} ds \left[\mu D\Phi(Z_s) + \frac{\sigma^2}{2} D^2\Phi(Z_s) \right]$$
$$+ \int_{(0,t]} D\Phi(Z_s) d(\rho - \lambda)_s + \sum_{n=1}^{N(t)} \Delta\Phi(Z)_{T_n}$$

It follows from (A.4) that $d\sum_{n=1}^{N(s)} \Delta\Phi(Z)_{T_n} = \sum_{n=1}^{\infty} \Delta\Phi(Z)_{T_n}\delta_{T_n}(ds)$. Integrating by parts (Elliott 1982, Corollary 12.22), we have

$$\begin{aligned} e^{-\alpha t}\Phi(Z_t) &= \int_{(0,t]} e^{-\alpha s} d\Phi(Z,p)_s - \alpha \int_{(0,t]} ds\, e^{-\alpha s}\Phi(Z_s) + \Phi(Z_0) \\ &= \Phi(Z_0) + \int_{(0,t]} e^{-\alpha s}\Bigg[\sigma D\Phi(Z_s)dW_s + ds\left(\mu D\Phi(Z_s) + \frac{\sigma^2}{2}D^2\Phi(Z_s)\right) \\ &\quad + D\Phi(Z_s)d(\rho-\lambda)_s + \sum_{n=1}^{\infty}\delta_{T_n}(ds)\Delta\Phi(Z)_{T_n}\Bigg] - \alpha\int_{(0,t]} ds\, e^{-\alpha s}\Phi(Z_s) \end{aligned}$$

Sorting terms on the right-hand-side, we have

$$\begin{aligned} e^{-\alpha t}\Phi(Z_t) &= \Phi(Z_0) + \sigma\int_{(0,t]} e^{-\alpha s} D\Phi(Z_s)dW_s + \int_{(0,t]} ds\, e^{-\alpha s}\Gamma\Phi(Z_s) \\ &\quad + \int_{(0,t]} e^{-\alpha s} D\Phi(Z_s)d(\rho-\lambda)_s + \int_{(0,t]}\left(\sum_{n=1}^{\infty}\Delta\Phi(Z)_{T_n}\delta_{T_n}\right)(ds)e^{-\alpha s} \end{aligned}$$

where the last term can be re-written as

$$\begin{aligned} \sum_{n=1}^{\infty}\Delta\Phi(Z)_{T_n}\int_{(0,t]}\delta_{T_n}(ds)e^{-\alpha s} &= \sum_{n=1}^{\infty}\Delta\Phi(Z)_{T_n}e^{-\alpha T_n}1_{(0,t]}(T_n) \\ &= \sum_{n=1}^{N(t)}\Delta\Phi(Z)_{T_n}e^{-\alpha T_n} \end{aligned}$$

Given our assumptions, it follows (Karatzas and Shreve 1988, Proposition 2.10) that the stochastic integral

$$\left(\int_{(0,t]} e^{-\alpha s} D\Phi(Z_s)dW_s\right)_{t\in\Re_+}$$

is a martingale. As $T_0 = 0$, it follows that $\Phi(Z_0) = \Phi(x) + \Delta\Phi(Z)_{T_0}$. Therefore, taking expectations in the above equation yields the result. ■

Proof of Lemma 2.3.2. Using (2.2.3) and Lemma 2.3.1, we have

$$\begin{aligned} &E\left[e^{-\alpha t}\Phi(Z_t) + C_t(L;x,\mu,\sigma,0,\delta,l-\delta,\alpha) - \Phi(x) - \int_{(0,t]} ds\, e^{-\alpha s}\Gamma\Phi(Z_s)\right] \\ &\quad = E\int_{(0,t]} e^{-\alpha s}[D\Phi(Z_s)+\delta]d\rho_s - E\int_{(0,t]} e^{-\alpha s}[D\Phi(Z_s)-(l-\delta)]d\lambda_s \\ &\quad\quad + E\sum_{n=0}^{N(t)} e^{-\alpha T_n}[\Phi(Z_{T_n}) - \Phi(Z_{T_n}-\Delta R_{T_n}) + \delta\Delta R_{T_n}] \\ &\quad\quad + E\sum_{n=0}^{N(t)} e^{-\alpha T_n}[\Phi(Z_{T_n}) - \Phi(Z_{T_n}+\Delta L_{T_n}) + (l-\delta)\Delta L_{T_n})] \end{aligned}$$

As

$$\Phi(Z_{T_n}) - \Phi(Z_{T_n} - \Delta R_{T_n}) + \delta \Delta R_{T_n} = \int_{Z_{T_n} - \Delta R_{T_n}}^{Z_{T_n}} dy \, [D\Phi(y) + \delta]$$

and

$$\Phi(Z_{T_n}) - \Phi(Z_{T_n} + \Delta L_{T_n}) + (l - \delta)\Delta L_{T_n}) = \int_{Z_{T_n}}^{Z_{T_n} + \Delta L_{T_n}} dy \, [D\Phi(y) - (l - \delta)]$$

our hypotheses imply that

$$E[e^{-\alpha t}\Phi(Z_t) + C_t(R, L; x, \mu, \sigma, 0, \delta, l - \delta, \alpha) - \Phi(x)] \geq 0 \qquad (A.5)$$

By the mean value theorem, $\Phi(Z_t) = \Phi(0) + D\Phi(c)Z_t$ for some $c \in [0, Z_t]$; consequently, $e^{-\alpha t}\Phi(Z_t) = e^{-\alpha t}\Phi(0) + e^{-\alpha t}D\Phi(c)Z_t$. Setting $\eta = \max\{\delta, |l - \delta|\}$, it follows from (a) that $|D\Phi(c)| \leq \eta$. As $Z_t \geq 0$, $|e^{-\alpha t}\Phi(Z_t)| \leq |e^{-\alpha t}\Phi(0)| + e^{-\alpha t}\eta Z_t$. Consequently, $Ee^{-\alpha t}|\Phi(Z_t)| \leq e^{-\alpha t}|\Phi(0)| + \eta E e^{-\alpha t} Z_t$. It follows from Definition 2.1.4(c) that $\lim_{t\uparrow\infty} Ee^{-\alpha t}|\Phi(Z_t)| = 0$. Thus, as $t \uparrow \infty$ the first term in (A.5) vanishes. ■

Proof of Lemma 2.3.8. Let $g(S) = e^{\beta S}[\gamma \cosh(\gamma S) - \beta \sinh(\gamma S)]$ for $S \in \Re_+$. Elementary calculations show that $Dg(S) = (\gamma^2 - \beta^2)e^{\beta S}\sinh(\gamma S) > 0$ for $S > 0$. Also, $D^2 g(S) = (\gamma^2 - \beta^2)e^{\beta S}[\gamma \cosh(\gamma S) + \beta \sinh(\gamma S)] > 0$ for $S \in \Re_+$.

Let $\delta > l$. Then, $g(0) = \gamma < \gamma\delta/(\delta - l)$. We show that there exists $S_0 > 0$ such that $g(S_0) \geq \gamma\delta/(\delta - l)$. Then the existence of a unique $S > 0$ solving (2.3.7) follows from the intermediate value theorem. If $g(1) < \gamma\delta/(\delta - l)$, then for $S > 1$, we have $g(S) = g(1) + \int_1^S du\, Dg(u) > g(1) + Dg(1)(S - 1)$. As $Dg(1) > 0$, $g(S)$ can be made arbitrarily large. Thus, there exists $S_0 > 0$ such that $g(S_0) > \gamma\delta/(\delta - l)$.

Conversely, suppose $\delta \leq l$. Clearly, if $\delta = l$, then (2.3.7) has no solution. If $\delta < l$, then (2.3.7) has no solution $S > 0$, as $g(0) = \gamma > 0 > \gamma\delta/(\delta - l)$ and g is increasing. ■

Proof of Lemma 2.3.9. (A) follows from construction.

(B) Clearly, these inequalities hold on the set $\{0\} \cup [S, \infty)$.

Consider $(0, S)$. Let $g(.; S) = DF(.; S)$. Then, $\Gamma g(x; S) = 0$ for every $x \in (0, S)$, $g(0; S) = -\delta < l - \delta = g(S; S) < 0$. If $g(x; S) < -\delta$ for some $x \in (0, S)$, then there exists $x^* \in (0, S)$ such that $g(x^*; S) = \min_{x \in [0,S]} g(x; S) < -\delta$. Consequently, $Dg(x^*; S) = 0$ and $D^2 g(x^*; S) \geq 0$. It follows that $\Gamma g(x^*; S) > 0$, which is a contradiction. Thus, $DF(x; S) = g(x; S) \geq -\delta$ for every $x \in (0, S)$.

We now show that $DF(x;S) = g(x;S) \leq l - \delta$ for every $x \in (0, S)$. Suppose there exists $x \in (0, S)$ such that $g(x;S) > l - \delta$. Then, there exists $x^* \in (0, S)$ such that $g(x^*;S) = \max_{x \in [0,S]} g(x;S)$. It follows that $Dg(x^*;S) = 0$. We also have $Dg(S;S) = D^2F(S;S) = D^2f(S;S) = 0$. Note that $\Gamma Dg(x;S) = 0$ for every $x \in (x^*, S)$. Suppose $\max_{x \in [x^*,S]} Dg(x;S) > 0$. Then, there exists $x^{**} \in (x^*, S)$ such that $\Gamma Dg(x^{**};S) < 0$, a contradiction. So, $Dg(x;S) \leq 0$ for every $x \in [x^*, S]$. Similarly, $Dg(x;S) \geq 0$ for every $x \in [x^*, S]$. So, $Dg(x;S) = 0$ for every $x \in [x^*, S]$. It follows that $l - \delta = g(S;S) = g(x^*;S) + \int_{x^*}^{S} dy\, Dg(y;S) = g(x^*;S) > l - \delta$, which is a contradiction.

(C) For $x \in [0, S]$, $\Gamma F(x;S) = \Gamma f(x;S) = 0$. Consider $x > S$. Note that, as $D^2f(S;S) = 0$, we have $\mu Df(S;S) - \alpha f(S;S) = \Gamma f(S;S) = 0$. Therefore,

$$\begin{aligned}
\Gamma F(x;S) &= \mu(l - \delta) - \alpha[f(S;S) + (l - \delta)(x - S)] \\
&= \mu(l - \delta) - \mu Df(S;S) - \alpha(l - \delta)(x - S) \\
&= -\alpha(l - \delta)(x - S) \\
&> 0
\end{aligned}$$

which concludes the proof. ■

Proof of Lemma 2.3.13. (A) Suppose (R, L) solves (2.3.12). It follows from (2.3.12) that $0 \leq Z_t = X_t + R_t - L_t \leq S$ for every $t \in \Re_+$. Consequently, conditions 2.1.4(a) and 2.1.4(c) are satisfied. Condition 2.1.4(b) will follow from the continuity of (R, L) and the conventions that $T_0 = 0$ and $\inf \emptyset = \infty$.

(i) It follows directly from (2.3.12) that R and L are non-negative and non-decreasing processes.

(ii) Consider $t \in \Re_+$. By (2.3.12), $R_t \geq L_t - X_t$ and $L_t \geq X_t + R_t - S$. If $R_t = L_t - X_t$, then $L_t > X_t + R_t - S$. Otherwise, $L_t = X_t + R_t - S$, which implies $S = 0$, a contradiction. Similarly, if $L_t = X_t + R_t - S$, then $R_t > L_t - X_t$.

(iii) If $R_0 = [L_0 - x]^+ > 0$, then $R_0 = L_0 - x$. By (ii), this means $L_0 > x + R_0 - S$. Since $L_0 = [x + R_0 - S]^+$, this means $L_0 = 0$. Therefore, $R_0 = -x \leq 0$, a contradiction. It follows that $R_0 = 0$ and $L_0 = [x - S]^+ \geq 0$.

(iv) Since, R and L are non-decreasing, their sample paths must have left-hand limits at every t, denoted by R_{t-} and L_{t-} respectively. Note that

$$R_{t-} = \lim_{n \uparrow \infty} R_{t-1/n} = \lim_{n \uparrow \infty} \sup \left\{ [L_u - X_u]^+ \mid u \in [0, t - 1/n] \right\}$$

Since

$$\sup\left\{[L_u - X_u]^+ \mid u \in [0, t-1/n]\right\} \le \sup\left\{[L_u - X_u]^+ \mid u \in [0, t)\right\}$$

for every $n \in \mathcal{Z}_{++}$, we have

$$R_{t-} = \lim_{n\uparrow\infty} \sup\left\{[L_u - X_u]^+ \mid u \in [0, t-1/n]\right\} \le \sup\left\{[L_u - X_u]^+ \mid u \in [0, t)\right\}$$

Conversely, for every $n \in \mathcal{Z}_{++}$, there exists $u_n \in (t-1/n, t)$ such that

$$\sup\left\{[L_u - X_u]^+ \mid u \in [0, t)\right\} - 1/n < [L_{u_n} - X_{u_n}]^+ \le R_{u_n}$$

Letting $n \uparrow \infty$, we have

$$\sup\left\{[L_u - X_u]^+ \mid u \in [0, t)\right\} \le R_{t-}$$

As an analogous argument applies to L_{t-}, we have

$$R_{t-} = \sup\left\{[L_u - X_u]^+ \mid u \in [0, t)\right\} \qquad \text{and} \qquad L_{t-} = \sup\left\{[X_u + R_u - S]^+ \mid u \in [0, t)\right\}$$

(v) Suppose $R_t > R_{t-}$ and $L_t > L_{t-}$ for some $t \in \Re_+$. Then, using (iv),

$$R_t = \sup\left\{[L_u - X_u]^+ \mid u \in [0, t]\right\} > \sup\left\{[L_u - X_u]^+ \mid u \in [0, t)\right\} = R_{t-}$$

and

$$L_t = \sup\left\{[X_u + R_u - S]^+ \mid u \in [0, t]\right\} > \sup\left\{[X_u + R_u - S]^+ \mid u \in [0, t)\right\} = L_{t-}$$

Consequently, $R_t = [L_t - X_t]^+ > 0$ and $L_t = [X_t + R_t - S]^+ > 0$. It follows that $R_t = L_t - X_t$ and $L_t = X_t + R_t - S$, which contradicts (ii). So, $R_t > R_{t-}$ implies $L_t = L_{t-}$, and similarly, $L_t > L_{t-}$ implies $R_t = R_{t-}$, i.e., R and L cannot jump at the same t.

(vi) Let $t \in \Re_+$ be such that $R_t - R_{t-} > 0$; by (v), this implies $L_t = L_{t-}$. Then, $R_t = [L_t - X_t]^+ > 0$. Consequently, $R_t = L_t - X_t$ and for every $n \in \mathcal{Z}_{++}$,

$$L_t - X_t = R_t > R_{t-} \ge [L_{t-1/n} - X_{t-1/n}]^+ \ge L_{t-1/n} - X_{t-1/n}$$

Therefore,

$$L_t - X_t > R_{t-} \ge \lim_{n\uparrow\infty}(L_{t-1/n} - X_{t-1/n}) = L_{t-} - X_{t-} = L_t - X_t$$

a contradiction. Thus, R is continuous on $\Re_+$. Similarly, L is continuous on $\Re_+$.

(B) We now define a control policy (R, L) that satisfies (2.3.12). Let $T_0 = 0$ and $(T_k)_{k\in\mathcal{Z}_{++}}$ be an increasing positive sequence of stopping times. Let

$$T_1 = \inf\{t > 0 \mid X_t - [x-S]^+ \leq 0 \quad \vee \quad X_t - [x-S]^+ \geq S\}$$

Since $X_t - [x-S]^+$ is a continuous process, $X_{T_1} - [x-S]^+ \in \{0, S\}$. If $X_{T_1} - [x-S]^+ = 0$, then define (R, L) as follows:

$$R_t = \begin{cases} 0, & \text{if } t \in [T_0, T_1) \\ R_{T_{2k}}, & \text{if } t \in [T_{2k}, T_{2k+1}) \\ \sup\{[L_u - X_u]^+ \mid u \in [0,t]\}, & \text{if } t \in [T_{2k-1}, T_{2k}) \end{cases} \tag{A.6a}$$

and

$$L_t = \begin{cases} [x-S]^+, & \text{if } t \in [T_0, T_1] \\ \sup\{[X_u + R_u - S]^+ \mid u \in [0,t]\}, & \text{if } t \in (T_{2k}, T_{2k+1}] \\ L_{T_{2k-1}}, & \text{if } t \in (T_{2k-1}, T_{2k}] \end{cases} \tag{A.6b}$$

If $X_{T_1} - [x-S]^+ = S$, then define (R, L) as follows:

$$R_t = \begin{cases} 0, & \text{if } t \in [T_0, T_1] \\ R_{T_{2k-1}}, & \text{if } t \in (T_{2k-1}, T_{2k}] \\ \sup\{[L_u - X_u]^+ \mid u \in [0,t]\}, & \text{if } t \in (T_{2k}, T_{2k+1}] \end{cases} \tag{A.7a}$$

and

$$L_t = \begin{cases} [x-S]^+, & \text{if } t \in [T_0, T_1] \\ \sup\{[X_u + R_u - S]^+ \mid u \in [0,t]\}, & \text{if } t \in (T_{2k-1}, T_{2k}] \\ L_{T_{2k-1}}, & \text{if } t \in (T_{2k}, T_{2k+1}] \end{cases} \tag{A.7b}$$

Given (R, L), define $Z_t = X_t + R_t - L_t$. We have already specified T_0 and T_1. Specify the other stopping times as follows: given $k \in \mathcal{Z}_{++}$, let

$$T_{2k} = \begin{cases} \inf\{t > T_{2k-1} \mid Z_t \geq S\}, & \text{if } Z_{T_1} = 0 \\ \inf\{t > T_{2k-1} \mid Z_t \leq 0\}, & \text{if } Z_{T_1} = S \end{cases} \tag{A.8a}$$

and

$$T_{2k+1} = \begin{cases} \inf\{t > T_{2k} \mid Z_t \leq 0\}, & \text{if } Z_{T_1} = 0 \\ \inf\{t > T_{2k} \mid Z_t \geq S\}, & \text{if } Z_{T_1} = S \end{cases} \tag{A.8b}$$

We now show that (R, L), defined by (A.6), (A.7) and (A.8), satisfies (2.3.12).

We first show that (2.3.12) holds for $t \in [0, T_1]$. Suppose there exists $t \in [0, T_1)$ such that $0 = R_t \neq \sup\{[L_u - X_u]^+ \mid u \in [0,t]\}$. It follows that, for some $u \in [0,t]$, $[L_u - X_u]^+ > 0$, i.e., $[x-S]^+ - X_u = L_u - X_u > 0$, but this contradicts the definition of T_1. Similarly, suppose there exists $t \in [0, T_1)$ such that $[x-S]^+ = L_t \neq$

$\sup\{[X_u + R_u - S]^+ \mid u \in [0,t]\} = \sup\{[X_u - S]^+ \mid u \in [0,t]\}$. It follows that, for some $u \in [0,t]$, $[X_u - S]^+ > [x - S]^+ \geq 0$, i.e. $X_u - S > [x-S]^+$, but this contradicts the definition of T_1.

We now show that (2.3.12) holds for intervals of the form $(T_{2k}, T_{2k+1}]$. For $t \in (T_{2k}, T_{2k+1}]$, (2.3.12b) holds by definition. Suppose there exists $t \in (T_{2k}, T_{2k+1}]$ such that (2.3.12a) does not hold, i.e., $R_{T_{2k}} = R_t \neq \sup\{[L_u - X_u]^+ \mid u \in [0,t]\}$. It follows from (A.6a) that $0 \leq R_{T_{2k}} < [L_u - X_u]^+$ for some $u \in (T_{2k}, t]$. It follows that $L_u - X_u = [L_u - X_u]^+ > R_{T_{2k}}$, which implies $X_u + R_u - L_u = X_u + R_{T_{2k}} - L_u < 0$, a contradiction of the definition of T_{2k+1}. An analogous proof can be given for intervals of the form $(T_{2k-1}, T_{2k}]$.

Finally, we show that the solution constructed above is unique. Suppose (R, L) and (R', L') are distinct solutions of (2.3.12). Let $\omega \in \Omega$ be such that $T(\omega) = \inf\{t \in \Re_+ \mid R_t(\omega) > R'_t(\omega)\} < \infty$. By definition, $R_t(\omega) = R'_t(\omega)$ for every $t \in [0, T(\omega))$. Consequently, $L_t(\omega) = L'_t(\omega)$ for every $t \in [0, T(\omega))$. By the continuity of $R(\omega)$ and $R'(\omega)$, this implies $R_{T(\omega)}(\omega) = R'_{T(\omega)}(\omega)$ and $L_{T(\omega)}(\omega) = L'_{T(\omega)}(\omega)$. Also, by continuity, there exists $\epsilon > 0$ such that $R_t(\omega) > R'_t(\omega)$ for every $t \in (T(\omega), T(\omega) + \epsilon)$. Consequently,

$$\begin{aligned} L_t(\omega) &= \sup\left\{[X_u(\omega) + R_u(\omega) - S]^+ \mid u \in [0,t]\right\} \\ &\geq \sup\left\{[X_u(\omega) + R'_u(\omega) - S]^+ \mid u \in [0,t]\right\} \\ &= L'_t(\omega) \end{aligned}$$

for every $t \in (T(\omega), T(\omega) + \epsilon)$. This means $R_{T(\omega)}(\omega) = L_{T(\omega)}(\omega) - X_{T(\omega)}(\omega)$. By (i), $L_{T(\omega)} > X_{T(\omega)}(\omega) + R'_{T(\omega)}(\omega) - S$. It follows that, for some $0 < \delta \leq \epsilon$, $L_t(\omega) = L_{T(\omega)}(\omega)$ for every $t \in (T(\omega), T(\omega) + \delta)$. Consequently, $L_t(\omega) \geq L'_t(\omega) \geq L'_{T(\omega)}(\omega) = L_{T(\omega)}(\omega) = L_t(\omega)$ for every $t \in (T(\omega), T(\omega) + \delta)$. Since $L_t(\omega) = L'_t(\omega)$ for every $t \in (T(\omega), T(\omega) + \delta)$, we have $R_t(\omega) = R'_t(\omega)$ for every $t \in (T(\omega), T(\omega) + \delta)$, a contradiction.

(C) By Lemma 2.3.1, the definition of C_t, and the definition of (R, L),

$$\begin{aligned} E[e^{-\alpha t} F(Z_t) &+ C_t(R, L; x, \mu, \sigma, 0, \delta, l - \delta, \alpha)] \\ &= F(x) + E\int_{(0,t]} e^{-\alpha s}[DF(Z_s) + \delta]\, d\rho_s \\ &\quad - E\int_{(0,t]} e^{-\alpha s}[DF(Z_s) - (l - \delta)]\, d\lambda_s + E\int_{(0,t]} ds\, e^{-\alpha s}\Gamma F(Z_s) \\ &\quad + E\Delta F(Z)_{T_0} + E(l - \delta)\Delta L_{T_0} \end{aligned} \tag{A.9}$$

Consider the right-hand-side of (A.9). By the definition of (R, L), $d\rho_s > 0$ if and only if $Z_s = 0$, and $d\lambda_s > 0$ if and only if $Z_s = S$. Since $DF(0) = -\delta$ and $DF(S) = l - \delta$, the second and third terms vanish. As $Z_s \in [0, S]$ for every $s > 0$, we have $\Gamma F(Z_s) = 0$ for every $s > 0$. Therefore, the fourth term vanishes. The last two terms can be written as $E[F(Z_0) - F(x) + (l - \delta)\Delta L_{T_0}]$. If $x \in [0, S]$, then $Z_0 = x$ and $\Delta L_{T_0} = 0$; consequently, $E[F(Z_0) - F(x) + (l - \delta)\Delta L_{T_0}] = 0$. If $x > S$, then $Z_0 = S$ and $\Delta L_{T_0} = x - S$; consequently, $E[F(Z_0) - F(x) + (l - \delta)\Delta L_{T_0}] = E[F(S) - F(x) + (l - \delta)(x - S)] = 0$. Thus, we have

$$E[e^{-\alpha t} F(Z_t) + C_t(R, L; x, \mu, \sigma, 0, \delta, l - \delta, \alpha)] = F(x)$$

By construction, $Z_t \in [0, S]$ for every $t \in \Re_+$. As $[0, S]$ is compact and F continuous, $\{F(Z_t) \mid t \in \Re_+\}$ is bounded. Therefore, as $t \uparrow \infty$, the first term vanishes, yielding the desired result. ■

Proof of Theorem 2.3.23. (A) Let $S(\delta, l)$ be defined by (2.3.7). Differentiating (2.3.7) with respect to δ and using (2.3.7) yields

$$\begin{aligned} D_1 S(\delta, l)(\delta - l)(\gamma^2 - \beta^2)e^{\beta S} \sinh(\gamma S) &= 2\gamma - e^{\beta S}[\gamma \cosh(\gamma S) - \beta \sinh(\gamma S)] \\ &= 2\gamma - \frac{2\gamma\delta}{\delta - l} \\ &= \frac{-2l\gamma}{\delta - l} \\ &< 0 \end{aligned}$$

Since $\delta > l$, we have $D_1 S(\delta, l) < 0$. Therefore, $D_1 S(h, l) = D_1 S(\delta, l)/\alpha < 0$.

Differentiating (2.3.7) with respect to l and using (2.3.7) yields

$$D_2 S(\delta, l)(l - \delta)^2(\gamma^2 - \beta^2)e^{\beta S} \sinh(\gamma S) = \gamma\delta > 0$$

Therefore, $D_2 S(\delta, l) > 0$.

(B) Let $C(x, \mu, \sigma, h, 0, l, \alpha) = EC_\infty(L; x, \mu, \sigma, h, 0, l, \alpha)$. If $\alpha' > \alpha > 0$, then L continues to be a feasible control policy and

$$\begin{aligned} C(x, \mu, \sigma, h, 0, l, \alpha) &= EC_\infty(R, L; x, \mu, \sigma, h, 0, l, \alpha) \\ &\geq EC_\infty(R, L; x, \mu, \sigma, h, 0, l, \alpha') \\ &\geq C(x, \mu, \sigma, h, 0, l, \alpha') \end{aligned}$$

i.e., C is decreasing in α. By analogous arguments, C is increasing in h and l.

(C) Given $t > 0$, let $C(x, \mu, \sigma, th, 0, tl, \alpha) = EC_\infty(L; x, \mu, \sigma, th, 0, tl, \alpha)$. Clearly,

$$EC_\infty(L; x, \mu, \sigma, th, 0, tl, \alpha) = tEC_\infty(L; x, \mu, \sigma, h, 0, l, \alpha) \geq tC(x, \mu, \sigma, h, 0, l, \alpha) \qquad (A.10)$$

By a similar argument, C is super-additive in (h, l):

$$C(x, \mu, \sigma, h + h', 0, l + l', \alpha) \geq C(x, \mu, \sigma, h, 0, l, \alpha) + C(x, \mu, \sigma, h', 0, l', \alpha) \qquad (A.11)$$

Combining (A.10) and (A.11), we see that C is concave in (h, l).

(D) We divide $\Re^2_{++}$ into three regions and consider each in turn.

(a) Let $(x, h) \in (0, S(h)) \times (l\alpha, \infty)$. By (2.3.6) and elementary calculations,

$$\lambda_1 Da(h) = \frac{e^{\beta S} - e^{\gamma S}}{2\alpha \sinh(\gamma S)} \qquad \text{and} \qquad \lambda_2 Db(h) = \frac{e^{-\gamma S} - e^{\beta S}}{2\alpha \sinh(\gamma S)} \qquad (A.12)$$

It follows from (2.3.4), (2.3.22), (A.12) and elementary calculations that

$$\begin{aligned} D_{12}c(x, h) &= 1/\alpha + \lambda_1 Da(h) e^{\lambda_1 x} + \lambda_2 Db(h) e^{\lambda_2 x} \\ &= \frac{1}{\alpha} + \frac{e^{\beta S} - e^{\gamma S}}{2\alpha \sinh(\gamma S)} e^{\lambda_1 x} + \frac{e^{-\gamma S} - e^{\beta S}}{2\alpha \sinh(\gamma S)} e^{\lambda_2 x} \\ &= \frac{1}{\alpha \sinh(\gamma S)} \left[\sinh(\gamma S) - e^{\beta(S-x)} \sinh(\gamma x) - e^{-\beta x} \sinh(\gamma(S - x)) \right] \end{aligned} \qquad (A.13)$$

It suffices to show that

$$e^{-\beta S} \sinh(\gamma S) - e^{-\beta x} \sinh(\gamma x) > e^{-\beta(S+x)} \sinh(\gamma(S - x))$$

for $x > 0$. Note that

$$\begin{aligned} e^{-\beta S} \sinh(\gamma S) - e^{-\beta x} \sinh(\gamma x) &= \int_x^S du\, e^{-\beta u} [\gamma \cosh(\gamma u) - \beta \sinh(\gamma u)] \\ &= \int_x^S du\, e^{-\beta(u+x)} e^{\beta x} [\gamma \cosh(\gamma u) - \beta \sinh(\gamma u)] \end{aligned}$$

Changing variables, $u = v + x$, yields

$$e^{-\beta S} \sinh(\gamma S) - e^{-\beta x} \sinh(\gamma x) = \int_0^{S-x} dv\, e^{-\beta(v+2x)} p(v, x)$$

where $p(v,x) = e^{\beta x}[\gamma \cosh(\gamma(v+x)) - \beta \sinh(\gamma(v+x))]$. Elementary calculation yields $D_2 p(v,x) = e^{\beta x}(\gamma^2 - \beta^2)\sinh(\gamma(v+x)) > 0$. Therefore, $p(.,x) > p(.,0)$ for every $x > 0$. This implies

$$\begin{aligned} e^{-\beta S}\sinh(\gamma S) - e^{-\beta x}\sinh(\gamma x) &> \int_0^{S-x} dv\, e^{-\beta(v+2x)} p(v,0) \\ &= \int_0^{S-x} dv\, e^{-\beta(v+2x)}[\gamma\cosh(\gamma v) - \beta\sinh(\gamma v)] \\ &= \int_0^{S-x} dv\, D_v[e^{-\beta(v+2x)}\sinh(\gamma v)] \\ &= e^{-\beta(S+x)}\sinh(\gamma(S-x)) \end{aligned}$$

(b) Let $(x,h) \in (S(h),\infty) \times (l\alpha,\infty)$. It follows from (2.3.4) and (2.3.22) that

$$c(x,h) = hx/\alpha + h\mu/\alpha^2 + f(S(h);S(h)) + (l - h/\alpha)(x - S(h))$$

As S is independent of x, we have $D_{12}c(x,h) = 0$.

(c) Let $(x,h) \in (0,\infty) \times (0, l\alpha)$. It follows from (2.3.4) and (2.3.22) that

$$c(x,h) = \frac{hx}{\alpha} + \frac{h\mu}{\alpha^2} + \frac{h}{\alpha(\beta+\gamma)} e^{-(\beta+\gamma)x}$$

By elementary calculations, $D_{12}c(x,h) = \left(1 - e^{-(\beta+\gamma)x}\right)/\alpha > 0$.

(E) We divide $\Re^2_{++}$ into three regions and consider each in turn.

(a) Let $(x,h) \in (0,S(h)) \times (l\alpha,\infty)$. From (A.12), we have $\lambda_1 Da(h) + \lambda_2 Db(h) = -1/\alpha$. Therefore, $\lambda_1 D^2 a(h) + \lambda_2 D^2 b(h) = 0$. It follows from (A.13) that

$$D_{122}c(x,h) = e^{\lambda_1 x}\lambda_1 D^2 a(h) + e^{\lambda_2 x}\lambda_2 D^2 b(h) = (e^{\lambda_1 x} - e^{\lambda_2 x})\lambda_1 D^2 a(h)$$

Clearly, $e^{\lambda_1 x} - e^{\lambda_2 x} < 0$. It follows from (A.12), (2.3.7) and (A) that

$$\begin{aligned} \lambda_1 D^2 a(h) &= \frac{DS(h)[e^{\beta S}(\beta\sinh(\gamma S) - \gamma\cosh(\gamma S)) + \gamma e^{\gamma S}(\cosh(\gamma S) - \sinh(\gamma S))]}{2\alpha\sinh^2(\gamma S)} \\ &= \frac{DS(h)}{2\alpha\sinh^2(\gamma S)}\left[\frac{\gamma\delta}{l-\delta} + \gamma\right] \\ &= \frac{DS(h) l\gamma}{2\alpha(l-\delta)\sinh^2(\gamma S)} \\ &> 0 \end{aligned}$$

as required.

(b) Let $(x,h) \in (S(h),\infty) \times (l\alpha,\infty)$. The result follows from part (b) of (D).

(c) Let $(x,h) \in (0,\infty) \times (0,l\alpha)$. The result follows from part (c) of (D).

(F) We divide $\Re^2_{++}$ into three regions and consider each in turn.

(a) Let $(x,h) \in (0,S(h)) \times (l\alpha,\infty)$. Suppose $\beta = 0$. Elementary calculation shows

$$D_{112}c(x,h) = \frac{\gamma\left[\cosh(\gamma(S(h)-x)) - \cosh(\gamma x)\right]}{\alpha\sinh(\gamma S(h))}$$

If $x < S(h)/2$, then $D_{112}c(x,h) > 0$, and if $x > S(h)/2$, then $D_{112}c(x,h) < 0$.

(b) Let $(x,h) \in (S(h),\infty) \times (l\alpha,\infty)$. The result follows from part (b) of (D).

(c) Let $(x,h) \in (0,\infty) \times (0,l\alpha)$. It follows from part (c) of (D) that $D_{112}c(x,h) = (\beta+\gamma)e^{-(\beta+\gamma)x}/\alpha > 0$. ■

Proof of Lemma 3.2.2. Suppose $U(x,x) = \sup\{U(x,x') \mid x' \in \Re_+\}$ for every $x \in \Re_+$. It follows that $D_2U(x,x) = 0$ for every $x \in \Re_+$. Let x, $x' \in \Re_+$ such that $x > x'$. Adding the inequalities $U(x,x) \geq U(x,x')$ and $U(x',x') \geq U(x',x)$, and cancelling common terms, we have $\int_{h(x)}^{h(x')} dh \int_{x'}^{x} dv\, D_{12}c(v,h) \geq 0$. Theorem 2.3.23(D) implies $\int_{x'}^{x} dv\, D_{12}c(v,h) \geq 0$. Therefore, $h(x') \geq h(x)$.

Conversely, suppose $D_2U(x,x) = 0$ for every $x \in \Re_+$ and h is non-increasing. From hypothesis and Theorem 2.3.23(D), it follows that $D_{12}U(y,v) = -D_{12}c(y,h(v))Dh(v) \geq 0$ for all y, $v \in \Re_+$. Therefore, for all x, $x' \in \Re_+$, we have

$$\begin{aligned} U(x,x) - U(x,x') &= \int_{x'}^{x} dv\, D_2U(x,v) \\ &= \int_{x'}^{x} dv\, [D_2U(x,v) - D_2U(v,v)] \\ &= \int_{x'}^{x} dv \int_{v}^{x} dy\, D_{12}U(y,v) \\ &\geq 0 \end{aligned}$$

which implies $U(x,x) = \sup\{U(x,x') \mid x' \in \Re_+\}$. ■

Proof of Theorem 3.4.14. (A) to (D) have been proved in the discussion of Regimes 1 and 2.

(E) By (D) and (E) of Theorem 2.3.23, $D_{12}c(x,h(x)) = 0$ and $D_{112}c(x,h(x)) = 0$. If $Dh(x) < 0$, then by (3.4.12), $Dh(x) = 0$, which is a contradiction.

(F) It follows from (E) that, if $Dh(x) < 0$, then $D_{12}c(x, h(x)) > 0$. Therefore,

$$D_2c(x, h(x)) + D\psi \circ h(x) = \frac{\lambda}{1+\lambda} G(x) D_{12}c(x, h(x)) > 0 = D_2c(x, h^*(x)) + D\psi \circ h^*(x)$$

By Assumption 3.1.2(d), $h(x) > h^*(x)$. ■

Proof of Theorem 3.4.15. We apply an extension of Filippov's existence theorem for Lagrange optimal control problems (Cesari 1983, Theorem 9.3.i). The extension (Cesari 1983, Section 9.5) replaces some compactness assumptions with weaker conditions.

Our proof verifies that the conditions of the extended Filippov's theorem are satisfied by our problem. We interpret the objects A, B, M, f_0, f and $\tilde{Q}$ that are used in the statement of Filippov's theorem, in the context of the above regulator's problem. In our problem,

$$(x, h(x), \mathcal{U}(x)) \in X \times [0, \bar{h}] \times \Re_+ \equiv A$$

$$(h(x), \mathcal{U}(x)) \in [0, \bar{h}] \times \Re_+ \equiv A(x), \quad \forall x \in X$$

$$(x(0), h \circ x(0), \mathcal{U} \circ x(0), x(1), h \circ x(1), \mathcal{U} \circ x(1)) \in \{x_0\} \times [0, \bar{h}] \times \{0\} \times \{x_1\} \times [0, \bar{h}] \times \Re_+ \equiv B$$

$$A \times \Re_+ \equiv M$$

$$-f(x) \left[V(x) + (1+\lambda)(v(x) - c(x,h) - \psi(h)) - \lambda\mathcal{U}\right] \equiv f_0(x; h, \mathcal{U}; y)$$

$$(-y, Dv(x) - D_1c(x, h)) \equiv f(x; h, \mathcal{U}; y)$$

$$\bigcup_{y \in \Re_+} \left\{(z^0, z) \in \Re^3 \mid z^0 \geq f_0(x; h, \mathcal{U}; y) \ \wedge \ z = f(x; h, \mathcal{U}; y)\right\} \equiv \tilde{Q}(x; h, \mathcal{U})$$

Clearly, B is closed, f_0 and f are continuous on M, and it is easily verified that $\tilde{Q}(x; h, \mathcal{U})$ is convex for every x and $(h, \mathcal{U}) \in A(x) = [0, \bar{h}] \times \Re_+$. We now check that A and M satisfy conditions (a), (b) and (c) stated in Section 9.5 of Cesari 1983. Note that A and M are closed and that $A \subset X \times \Re^2$.

(a) is satisfied immediately as M is closed and every closed ball of finite radius in $\Re^4$ is compact.

(b) Set $P = X \times [0, \bar{h}] \times \{0\}$. P is a compact subset of A and every admissible trajectory must intersect P.

(c) Note that $h(-y) + \mathcal{U}[Dv(x) - D_1c(x,h)] \leq \mathcal{U}[Dv(x) - D_1c(x,h)] \leq C(h^2 + \mathcal{U}^2 + 1)$ for every $(x; h, \mathcal{U}; y) \in X \times [0, \bar{h}] \times \Re_+ \times \Re_+$, where $C = \max\{Dv(x) - D_1c(x,h) \mid (x,h) \in X \times [0, \bar{h}]\}$; C exists as $X \times [0, \bar{h}]$ is compact and $Dv - D_1c$ is continuous. ■

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