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Best reply versus dominance

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Abstract

The central results of this paper are dualities between actions in a decision problem that are not strongly (resp., weakly) dominated over a state space and actions that are best (resp., internal-best) replies to a state. The results admit action and state spaces that are subsets of abstract topological vector spaces. The generality of this setting significantly expands the set of applications of the dualities in comparison to their predecessors. This is demonstrated in the game-theoretic setting by applying the dualities to a player's decision problem in an abstract many-player game as well as in the mixed extension of a many-player game. The formalism also allows applications beyond the game-theoretic setting, such as the characterisation of various welfare-theoretic notions of efficient outcomes in terms of the best reply properties of the outcomes.

JEL classification: C72, D81

Key words: duality, best reply, internal-best reply, strong dominance, weak dominance, efficiency

1 Introduction

This paper concerns decision problems (X, Y, u) wherein X is the decision-maker's action space, Y is the state space, and $u : X \times Y \to \Re$ yields $u(x, y)$ as the decision-maker's utility from action x in state y.

In Section 2, we shall define the notion of an action being strongly (resp., weakly) dominated by another action over the state space and the notion of an action being a best (resp., an internal-best) reply to a state.[1] Our central results are duality theorems. The first result, Theorem 2.2, provides conditions ensuring that an action is not strongly dominated if and only if it is a best reply. The second result, Theorem 2.3, states conditions implying

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[1] An internal-best reply generalises the notion of a cautious response in Pearce [10].

that an action is not weakly dominated if and only if it is an internal-best reply. The action space and the state space in these results are subsets of abstract topological vector spaces. This generality significantly enhances the applicability of these results in comparison to their predecessors.

In Section 3, we gather most of the technical prerequisites for the proofs of the main results and their applications.

We prove the main results in Section 4. The proof of Theorem 2.2 is a direct application of Browder's coincidence theorem, which is stated as Lemma 3.1. The proof of Theorem 2.3 requires a longer sequence of arguments that use Theorem 2.2 to relate weak domination and internal-best replies *via* the intermediate notion of serial domination.

Section 5 features applications of the central results, starting with game-theoretic applications and ending with welfare-theoretic applications. Each game-theoretic application implies that two well-founded but seemingly different methods of pruning the set of predictions about a game will yield identical results. Each welfare-theoretic application shows that an efficient outcome is one that maximises an appropriate welfare criterion.

In Section 5.1, we interpret X as a player's strategy space, Y as the space of the opponents' strategy profiles, and u as the given player's payoff function. Both our central results are applicable to such decision problems in many-player games (i.e., those with two or more players) when the games satisfy appropriate assumptions. Application of Theorem 2.2 (resp., 2.3) yields Theorem 5.1 (resp., 5.2).

In Section 5.2, we apply Theorem 2.2 to a player's decision problem in the mixed extension of a game. This involves interpreting X as the given player's space of mixed strategies, Y as the space of profiles of the opponents' mixed strategies, and u as the given player's expected payoff function. This leads to Theorem 5.4, which states the duality between a mixed strategy's strong dominance property and its best reply property.

In Section 5.3, we stay with the mixed extension of a game and apply Theorem 2.3 to a player's decision problem. This results in Theorems 5.7 and 5.10. The former result derives the duality between a mixed strategy's weak dominance property and its internal-best reply property when it is in the interior of the player's space of mixed strategies. The latter result derives the analogous duality for frontier points of the player's space of mixed strategies that meet a relative interiority condition.

The mixed extension of a two-player game is the setting for all earlier versions of our duality results. The earliest versions considered problems of the form $(\Delta(C_1), \Delta(C_2), u)$ wherein $\Delta(C_1)$ and $\Delta(C_2)$ are the sets of mixed strategies over finite sets of pure strategies C_1 and C_2 respectively, and $u(x, y)$ is the expected utility generated by a von Neumann-Morgenstern utility defined on $C_1 \times C_2$ and the probability measure $x \times y$ derived from $(x, y) \in \Delta(C_1) \times \Delta(C_2)$. The genealogy of these results includes van Damme [4], Gale and Sherman [7], and Pearce [10].[2]

Zimper [12] generalises the duality between the strong dominance property and the best reply property by admitting C_1 and C_2 that are compact metric spaces instead of finite sets. Theorem 2.2 further generalises this result to the specified abstract setting. Theorems 5.1 and 5.4 are implications of this abstract result in different settings. With respect to the duality between the weak dominance property and the internal-best reply property, there do not seem to be earlier results in the literature with the scope and generality of Theorem 2.3 and its implications stated as Theorems 5.2, 5.7, and 5.10.

It is well-known that the dualities for mixed extensions do not extend generally from two-player games to many-player games.[3] Our abstract results suggest the reason for this failure: in a many-player game, the decision-maker's expected utility generally is not an affine function of the opponents' mixed strategy profile, and therefore it cannot generally satisfy the hypotheses underlying the duality results. However, we specify a class of von Neumann-Morgenstern utilities for which this problem does not arise in the mixed extension of many-player games. Moreover, this specification involves no loss of generality in the mixed extension of a two-player game, which is the setting for the earlier results in the literature.

In Section 5.4, we apply the main results to welfare economics. The derived results are not game-theoretic in nature and do not involve probabilistic constructs. The notions of dominance are used to define weak and strong forms of Pareto and Utilitarian efficient outcomes. The duality results imply that an outcome is efficient in terms of these concepts if and only if it is optimal with respect to an appropriate welfare criterion. These characterisations may be used to identify various kinds of efficient outcomes in multi-agent allocation models, including those with infinite-dimensional outcome spaces.

Finally, we conclude the paper in Section 6 with a résumé of our results.

2 Main results

We shall study the relationships among the notions defined below.

Definition 2.1 *Given* (X, Y, u), $x_0 \in X$ *is said to be*

(a) *strongly dominated if* $u(x, .) > u(x_0, .)$ *on* Y *for some* $x \in X$,

[2]The special version of Theorem 2.2 (resp., 2.3) in this setting concerns systems of linear inequalities and is an immediate corollary of Tucker's (resp., Motzkin's) theorem of the alternative, which is stated as Theorem 2.4.3 (resp., 2.4.2) in Mangasarian [8].

[3]Although the setting in Zimper [12] is *prima facie* a many-player game, the assumed ability of the decision-maker's opponents to correlate their strategy choices effectively reduces the model to a two-player game.

(b) weakly dominated if $u(x,.) \geq u(x_0,.)$ *and* $u(x,.) \neq u(x_0,.)$ *on* Y *for some* $x \in X$,

(c) a best reply if $u(x_0, y_0) \geq u(., y_0)$ *on* X *for some* $y_0 \in Y$, *and*

(d) an internal-best reply if Y *is a subset of a vector space and* $u(x_0, y_0) \geq u(., y_0)$ *on* X *for some* y_0 *that is an internal point of* Y*; internal points are defined and discussed in Section 3.*

If X and Y are nonempty, X is compact, and u is continuous, then there is a best reply, which cannot be strongly dominated. If, in addition, Y is a subset of a vector space and has an internal point, then there exists an internal-best reply. Lemma A.1 in Appendix A provides conditions for the existence of an action that is not weakly dominated. Our first duality result relates the strong dominance property and the best reply property.

Theorem 2.2 *Suppose* (X, Y, u) *satisfies the following hypotheses:*

(a) X *is a convex subset of a topological vector space,*

(b) Y *is a nonempty, convex, and compact subset of a locally convex topological vector space,*

(c) u *is continuous,*

(d) $u(x, ty + (1-t)y') = tu(x, y) + (1-t)u(x, y')$ *for all* $x \in X$, $y, y' \in Y$, *and* $t \in (0, 1)$, *and*

(e) $u(., y)$ *is quasi-concave for every* $y \in Y$.

Then, $x_0 \in X$ *is not strongly dominated if and only if it is a best reply.*

The second duality result relates the weak dominance property and the internal-best reply property.

Theorem 2.3 *Suppose* (X, Y, u) *satisfies the following hypotheses:*

(a) X *is a convex, compact, and metrisable subset of a Hausdorff locally convex topological vector space,*

(b) Y *is a convex, compact, and metrisable subset of a locally convex topological vector space that contains a nonempty set of internal points,*

(c) u *is continuous,*

(d) $u(x, ty + (1-t)y') = tu(x, y) + (1-t)u(x, y')$ *for all* $x \in X$, $y, y' \in Y$, *and* $t \in (0, 1)$, *and*

(e) $u(tx + (1-t)x', y) = tu(x, y) + (1-t)u(x', y)$ *for all* $x, x' \in X$, $y \in Y$, *and* $t \in (0, 1)$.

Suppose $x_0 \in X$ *is such that* $u(x, .) \neq u(x_0, .)$ *for every* $x \in X \setminus \{x_0\}$.

(A) *If* x_0 *is an interior point of* X, *then* x_0 *is weakly dominated if and only if it is not an internal-best reply.*

(B) *If* x_0 *is a frontier point of* X *and there is a compact set* $C \subset X \setminus \{x_0\}$ *such that, for every* $x \in X \setminus \{x_0\}$, $x_0 + t(x - x_0) \in C$ *for some* $t \geq 1$, *then* x_0 *is weakly dominated if and only if it is not an internal-best reply.*

Evidently, the hypotheses of this result are stronger than those of Theorem 2.2. In an application, the requirement that the state space must have internal points may be met by (a) showing that it has interior points relative to an appropriate ambient topological vector space, and (b) using the equivalence of interior and internal points in this case.

We defer the proofs of the above results until Section 4. Meanwhile, we gather some technical preliminaries in the next section.

3 Preliminaries

While many notational conventions and technical facts will be stated as and when required, the following conventions will be in force throughout this paper. $\mathcal{N}$ denotes the set of natural numbers. Given sets A and B, $A \setminus B$ is their set difference. If A and B are nonempty subsets of a vector space, then $A - B$ is their algebraic difference. Given a mapping $F : X \to 2^Y$, $F(Z) \equiv \cup_{z \in Z} F(z)$ for $Z \subset X$ and $F^{-1}(y) \equiv \{x \in X \mid y \in F(x)\}$ for $y \in Y$.

Given a topological space, the interior of a subset Z is $\text{Int}\, Z$ and its frontier is $\text{Fr}\, Z$. Unless notified otherwise, subsets of topological spaces will have the subspace topology, products of topological spaces will have the product topology, and the real line $\Re$ will have the Euclidean topology.

The following coincidence theorem (Browder [3], Theorem 7) is the key tool used in the proof of Theorem 2.2.

Lemma 3.1 *Suppose* K *is a nonempty and convex subset of a topological vector space* $\mathcal{E}$ *and* K_1 *is a nonempty, compact, and convex subset of a locally convex topological vector space* $\mathcal{F}$. *If* $T : K \to 2^{K_1}$ *and* $S : K \to 2^{K_1}$ *are such that*

(a) T *has a closed graph and nonempty convex values,*

(b) $S(u)$ *is an open subset of* K_1 *for every* $u \in K$, *and*

(c) $S^{-1}(v)$ *is nonempty and convex for every* $v \in K_1$,

then there exists $u_0 \in K$ *such that* $T(u_0) \cap S(u_0) \neq \emptyset$.

Given a vector space $\mathcal{K}$ with origin $0_{\mathcal{K}}$ and $K \subset \mathcal{K}$, $x \in \mathcal{K}$ is called an internal point of K if, for every $z \in \mathcal{K}$, there exists $\epsilon > 0$ such that $|\delta| < \epsilon$ implies $x + \delta z \in K$. Let K^* be the set of internal points of K, which is called the core or algebraic interior of K.

If K is convex and $0_{\mathcal{K}} \in K^*$, then the support function of K is $f : \mathcal{K} \to \Re$, defined by $f(x) = \inf\{a > 0 \mid x/a \in K\}$ for $x \in \mathcal{K}$. It can be shown that $f(x) \in \Re_+$; $f(\alpha x) = \alpha f(x)$ for $\alpha \geq 0$; $f(x) \leq 1$ for $x \in K$; $f(x + y) \leq f(x) + f(y)$ for $x, y \in \mathcal{K}$; and $f(x) < 1$ if and only if $x \in K^*$ (Dunford and Schwartz [6], Lemma V.1.8).

If $\mathcal{K}$ is a topological vector space, then $\operatorname{Int} K \subset K^*$, and if K is convex and $\operatorname{Int} K \neq \emptyset$, then $\operatorname{Int} K = K^*$ (Dunford and Schwartz [6], Theorem V.2.1). The following approximation result is proved in Appendix A.

Lemma 3.2 *If* K *is a convex and metrisable subset of a topological vector space* $\mathcal{K}$ *and* $K^* \neq \emptyset$, *then* K^* *is dense in* K.

The notion of a set's internal approximation, as defined below, is essential for defining the concept of serial domination as *per* Definition 4.2.

Definition 3.3 *Let* $\mathcal{Y}$ *be a topological vector space.* $Y \subset \mathcal{Y}$ *is said to be internally approximated by a family of sets* $\{K_n \mid n \in \mathcal{N}\}$ *if*

(a) *every* K_n *is nonempty, convex, and compact,*

(b) $K_n \subset K_{n+1}$ *for every* $n \in \mathcal{N}$, *and*

(c) $Y^* = \cup_{n \in \mathcal{N}} K_n$.

Next, we provide sufficient conditions for a set to be internally approximable; the proof is in Appendix A.

Lemma 3.4 *If* Y *is a convex and compact subset of a topological vector space* $\mathcal{Y}$ *and* $Y^* \neq \emptyset$, *then there exists a family of sets* $\{K_n \mid n \in \mathcal{N}\}$ *that internally approximates* Y.

Finally, we note some facts regarding spaces of measures that will be used in Section 5. Consider a compact metric space T. Let $\mathcal{B}(T)$ be the Borel σ-algebra on T. Let $\operatorname{rca}(T)$, given the total variation norm $\|.\|$, be the Banach space of regular countably additive Borel measures on $(T, \mathcal{B}(T))$. Let $\mathcal{C}(T)$, given the supremum norm, be the Banach space of continuous and bounded real-valued functions on T. By the Riesz representation theorem (Dunford and Schwartz [6], Theorem IV.6.3), the dual space of $C(T)$ is isometrically isomorphic to $\text{rca}(T)$. So, the $\mathcal{C}(T)$ topology of $\text{rca}(T)$ is its weak* topology.[4]

Consider $\text{rca}(T)$ with the weak* topology. Then, $\text{rca}(T)$ is a Hausdorff locally convex topological vector space (Dunford and Schwartz [6], Lemma V.3.3).[5] As T is compact, $\mathcal{C}(T)$ is separable (Aliprantis and Border [1], Lemma 3.99). Consequently, $B_r \equiv \{x \in \text{rca}(T) \mid \|x\| \leq r\}$ is metrisable for $r > 0$ (Dunford and Schwartz [6], Theorem V.5.1).

Let $\Delta(T)$ be the set of probability measures on $(T, \mathcal{B}(T))$. As $\Delta(T) \subset \text{rca}(T)$ (Parthasarathy [9], Theorem II.1.2) and $\|x\| = 1$ for every $x \in \Delta(T)$, we have $\Delta(T) \subset B_1$. Moreover, $\Delta(T)$ is convex, compact, and metrisable (Parthasarathy [9], Theorem II.6.4), and therefore separable.

4 Proofs

We start with the duality between strong dominance and best replies. In addition to being important in its own right, this result will play a crucial role in the proof of Theorem 2.3 *via* its application in Lemma 4.3.

Proof of Theorem 2.2 Suppose x_0 is strongly dominated and is a best reply. Then, there exists $x \in X$ such that $u(x,.) > u(x_0,.)$ on Y and there exists $y \in Y$ such that $u(x_0, y) \geq u(., y)$ on X. So, $u(x, y) > u(x_0, y) \geq u(x, y)$, a contradiction.

Conversely, suppose x_0 is not strongly dominated and not a best reply. We shall use Lemma 3.1, setting $K \equiv X$ and $K_1 \equiv Y$. $X \neq \emptyset$ as $x_0 \in X$. The other requirements of K and K_1 are satisfied by hypotheses (a) and (b). Define $T : X \to 2^Y$ by $T(x) = \{y \in Y \mid u(x_0, y) \geq u(x, y)\}$ and $S : X \to 2^Y$ by $S(x) = \{y \in Y \mid u(x_0, y) < u(x, y)\}$. As u is continuous by hypothesis (c), $\text{Gr}\, T = \{(x, y) \in X \times Y \mid u(x_0, y) \geq u(x, y)\}$ is closed in $X \times Y$ and $S(x)$ is open in Y for every $x \in X$. As x_0 is not strongly dominated, $T(x) \neq \emptyset$ for every $x \in X$. For every $x \in X$, as $u(x,.)$ is linear on Y by hypothesis (d), $T(x)$ is convex. For every $y \in Y$, $S^{-1}(y) = \{x \in X \mid u(x_0, y) < u(x, y)\} \neq \emptyset$ as x_0 is not a best reply and $S^{-1}(y)$ is convex as $u(., y)$ is quasi-concave by hypothesis (e). By Lemma 3.1, $T(x) \cap S(x) \neq \emptyset$ for some $x \in X$, a contradiction. ■

Next, we turn to the duality between weak dominance and internal-best replies as stated in Theorem 2.3. The proof involves bridging the dual notions *via* the intermediate notion of a decision-maker's action being serially dominated. The detailed plan for doing this is as follows:

1. Lemma 4.1 shows that, given $Y^* \neq \emptyset$ and the hypotheses of Theorem 2.2, an action is weakly dominated on Y if and only if it is

[4] This is the weakest topology on $\text{rca}(T)$ that makes $x \mapsto \int_T x(dt)\, f(t)$ continuous for each $f \in \mathcal{C}(T)$, i.e., it is the projective topology generated by these linear functionals.

[5] The space is Hausdorff as the relevant family of linear functionals is total.

strongly dominated on Y^*.

2. Using the internal approximation of Y by a family of sets $\{K_n \mid n \in \mathcal{N}\}$, Definition 4.2 formalises the notion of action x_0 being serially dominated by a sequence of actions (x_n), where each x_n strongly dominates x_0 on K_n. Since each $K_n \subset Y^*$, Lemma 4.1 implies that, if an action is weakly dominated on Y, then it is serially dominated.

3. Lemma 4.3 shows that, given the hypotheses of Theorem 2.2, x_0 is an internal-best reply if and only if it is not serially dominated. Combining this fact with Step 2, if $Y^* \neq \emptyset$ and an action is weakly dominated on Y, then it is not an internal-best reply. The converse requires the demonstration that, if an action is serially dominated, then it is weakly dominated on Y.

4. Lemma 4.4 provides sufficient conditions for an action to be weakly dominated if and only if it is serially dominated.

5. Theorems 2.3(A) and (B) follow from Lemmas 4.3 and 4.4.

We proceed to the first step of the plan outlined above.

Lemma 4.1 *Suppose (X, Y, u) satisfies the hypotheses of Theorem 2.2. If Y is metrisable and $Y^* \neq \emptyset$, then $x_0 \in X$ is weakly dominated if and only if there exists $x \in X$ such that $u(x_0, .) < u(x, .)$ on Y^*.*

Proof. Suppose x_0 is weakly dominated by $x \in X$. Then, there exists $y_0 \in Y$ such that $u(x_0, y_0) < u(x, y_0)$. Consider $y \in Y^*$. As $y - y_0 \in \mathcal{Y}$, there exists $\epsilon > 0$ such that $y + \delta(y - y_0) \in Y$ for every $\delta \in (0, \epsilon)$. Let $\delta_0 \in (0, \epsilon)$. Then, $y_1 \equiv y + \delta_0(y - y_0) \in Y$ and $y = ty_1 + (1-t)y_0$, where $t = (1 + \delta_0)^{-1} \in (0, 1)$. As $u(x_0, y_1) \leq u(x, y_1)$, hypothesis (d) implies $u(x_0, y) = u(x_0, ty_1 + (1-t)y_0) = tu(x_0, y_1) + (1-t)u(x_0, y_0) < tu(x, y_1) + (1-t)u(x, y_0) = u(x, ty_1 + (1-t)y_0) = u(x, y)$. So, $u(x_0, .) < u(x, .)$ on Y^*.

Conversely, suppose there exists $x \in X$ such that $u(x_0, .) < u(x, .)$ on Y^*. Consider $y \in Y$. As Y is metrisable and $Y^* \neq \emptyset$, combining Lemma 3.2 and hypothesis (b), we conclude that Y^* is dense in Y. So, there is a sequence $(y_n) \subset Y^*$ converging to y. As $u(x_0, y_n) < u(x, y_n)$ for every n and u is continuous by hypothesis (c), we have $u(x_0, y) \leq u(x, y)$. As $Y^* \neq \emptyset$, there exists $y_0 \in Y^*$ and $u(x_0, y_0) < u(x, y_0)$. So, x_0 is weakly dominated. ∎

We are ready for the second step of defining serial domination.

Definition 4.2 *Consider (X, Y, u). $x_0 \in X$ is said to be serially dominated if, for every family of sets $\{K_n \mid n \in \mathcal{N}\}$ that internally approximates Y, and for every $n \in \mathcal{N}$, there exists $x \in X$ such that $u(x_0, .) < u(x, .)$ on K_n.*

The next step establishes the duality between serial domination and internal-best replies.

Lemma 4.3 *If (X, Y, u) satisfies the hypotheses of Theorem 2.2, then $x_0 \in X$ is not serially dominated if and only if it is an internal-best reply.*

Proof. If $Y^* = \emptyset$, then the result holds trivially as x_0 is not an internal-best reply and x_0 is serially dominated because there is no family of sets that internally approximates Y. Henceforth, let $Y^* \neq \emptyset$.

Suppose x_0 is not serially dominated for (X, Y, u). Then, there exists a family of sets $\{K_n \mid n \in \mathcal{N}\}$ that internally approximates Y, and there exists $n \in \mathcal{N}$ such that, for every $x \in X$, $u(x_0, y) \geq u(x, y)$ for some $y \in K_n$. Using Definition 3.3, (X, K_n, u) satisfies the hypotheses of Theorem 2.2. Therefore, x_0 is not strongly dominated with respect to (X, K_n, u). By Theorem 2.2, x_0 is a best reply with respect to (X, K_n, u), i.e., $u(x_0, y) \geq u(., y)$ on X for some $y \in K_n$. As $y \in K_n \subset Y^*$, x_0 is an internal-best reply for (X, Y, u).

Conversely, suppose x_0 is an internal-best reply and it is serially dominated with respect to (X, Y, u). As x_0 is an internal-best reply, $u(x_0, y) \geq u(., y)$ on X for some $y \in Y^*$. As $Y^* \neq \emptyset$, Lemma 3.4 and hypothesis (b) imply that there is a family of sets $\{K_n \mid n \in \mathcal{N}\}$ that internally approximates Y. As x_0 is serially dominated, for every $n \in \mathcal{N}$, there exists $x_n \in X$ such that $u(x_0, .) < u(x_n, .)$ on K_n. As $y \in Y^* = \cup_{n \in \mathcal{N}} K_n$, it follows that $y \in K_n$ for some $n \in \mathcal{N}$. Consequently, $u(x_n, y) \leq u(x_0, y) < u(x_n, y)$, a contradiction. ■

The fourth step provides conditions for the equivalence of serial domination and weak domination.

Lemma 4.4 *Consider (X, Y, u) and $x_0 \in X$ that satisfy the hypotheses of Theorem 2.3. If there is a compact set $C \subset X \setminus \{x_0\}$ such that, for every $x \in X \setminus \{x_0\}$, $h(t) \equiv x_0 + t(x - x_0) \in C$ for some $t \geq 1$, then x_0 is weakly dominated if and only if it is serially dominated.*

Proof. Suppose x_0 is weakly dominated. By Lemma 4.1, there exists $x \in X$ such that $u(x_0, .) < u(x, .)$ on Y^*. Consider a family of sets $\{K_n \mid n \in \mathcal{N}\}$ that internally approximates Y. For every $n \in \mathcal{N}$, as $K_n \subset Y^*$, we have $u(x_0, .) < u(x, .)$ on K_n. So, x_0 is serially dominated.

Conversely, suppose x_0 is not weakly dominated. We show that x_0 is not serially dominated.

1. Consider $x \in C$. Then, $x \neq x_0$. If $u(x_0, .) \leq u(x, .)$ on Y, then $u(x, .) \neq u(x_0, .)$ implies $u(x_0, y) < u(x, y)$ for some $y \in Y$. Thus, x_0 is weakly dominated, which is a contradiction. So, $u(x_0, y) > u(x, y)$ for some $y \in Y$. As Y^* is dense in Y by Lemma 3.2 and u is continuous by hypothesis (c), $u(x_0, y(x)) > u(x, y(x))$ for some $y(x) \in Y^*$. As u

is continuous, there is an open neighbourhood $V(x)$ of x such that $u(x_0, y(x)) > u(., y(x))$ on $V(x) \cap X$.

The collection $\mathcal{V} \equiv \{V(x') \mid x' \in C\}$ is an open cover of C. As C is compact, $\mathcal{V}$ contains a finite subcover of C, say $\{V(x_1), \ldots, V(x_n)\}$ for some $\{x_1, \ldots, x_n\} \subset C$.

It follows that $x \in V(x_j)$ for some $j \in \{1, \ldots n\}$. Since $u(x_0, y(x_j)) > u(., y(x_j))$ on $V(x_j) \cap X$, we have $u(x_0, y(x_j)) > u(x, y(x_j))$.

Thus, for every $x \in C$, there exists $y_x \in \{y(x_1), \ldots, y(x_n)\} \subset Y^*$ such that $u(x_0, y_x) > u(x, y_x)$.

2. Let $x \in X \setminus \{x_0\}$. By assumption, $h(t) \in C$ for some $t \geq 1$. By step 1, there exists $y_{h(t)} \in \{y(x_1), \ldots, y(x_n)\}$ such that $u(x_0, y_{h(t)}) > u(h(t), y_{h(t)})$. As $t \geq 1$, $x = t^{-1}h(t) + (1 - t^{-1})x_0$ and $u(x, y_{h(t)}) = (1 - t^{-1})u(x_0, y_{h(t)}) + t^{-1}u(h(t), y_{h(t)}) < u(x_0, y_{h(t)})$.

   Thus, for every $x \in X \setminus \{x_0\}$, there exists $y \in \{y(x_1), \ldots, y(x_n)\} \subset Y^*$ such that $u(x_0, y) > u(x, y)$.

3. By hypothesis (b) and Lemma 3.4, there is a family of sets $\{K_n \mid n \in \mathcal{N}\}$ that internally approximates Y. So, $\{y(x_1), \ldots, y(x_n)\} \subset Y^* = \cup_{n \in \mathcal{N}} K_n$ and $K_n \subset K_{n+1}$ for every $n \in \mathcal{N}$.

   Thus, $\{y(x_1), \ldots, y(x_n)\} \subset K_m$ for some $m \in \mathcal{N}$.

4. Suppose x_0 is serially dominated. Then, there exists $x^* \in X$ such that $u(x_0, .) < u(x^*, .)$ on K_m. So, $x^* \in X \setminus \{x_0\}$. By steps 2 and 3, there exists $y^* \in \{y(x_1), \ldots, y(x_n)\} \subset K_m$ such that $u(x^*, y^*) < u(x_0, y^*)$. As $y^* \in K_m$, we have $u(x^*, y^*) < u(x_0, y^*) < u(x^*, y^*)$, which is a contradiction. ■

Finally, we arrive at the proof of the duality between weak domination and internal-best replies.

Proof of Theorem 2.3 (A) As X is compact and $\mathcal{X}$ is Hausdorff by hypothesis (a), X is closed in $\mathcal{X}$. Since $x_0 \in \operatorname{Int} X$, we have $C \equiv \operatorname{Fr} X \subset X \setminus \{x_0\}$. As C is a closed subset of the compact set X, C is compact. We show that, for every $x \in X \setminus \{x_0\}$, $x_0 + t(x - x_0) \in C$ for some $t \geq 1$. Then, the result follows from Lemmas 4.3 and 4.4.

Consider $x \in X \setminus \{x_0\}$. Define $h : [1, \infty) \to \mathcal{X}$ by $h(t) = x_0 + t(x - x_0)$ and $g : [1, \infty) \to \mathcal{X}$ by $g(t) = t(x - x_0)$. Then, $h(.) = x_0 + g(.)$, $h^{-1}(X) = g^{-1}(X - \{x_0\})$, and $1 \in h^{-1}(X)$. As $\mathcal{X}$ is a topological vector space by hypothesis (a), h is continuous. So, $h^{-1}(X)$ is closed in $[1, \infty)$. Suppose $g^{-1}(X - \{x_0\})$ is bounded. Then, $h^{-1}(X)$ is nonempty and compact, and there exists $t = \max h^{-1}(X)$. As $h(t) \in X$ and $h(t + \epsilon) \notin X$ for every $\epsilon > 0$, we have $h(t) \in \operatorname{Fr} X = C$.

It remains to show that $g^{-1}(X - \{x_0\})$ is bounded. Suppose it is not bounded. Then, there exists an unbounded, strictly increasing sequence $(t_n) \subset [1, \infty)$ such that $g(t_n) = t_n(x - x_0) \in X - \{x_0\}$ for every $n \in \mathcal{N}$.

As $\mathcal{X}$ is a Hausdorff locally convex topological vector space by hypothesis (a), it is endowed with the projective topology generated by a total family of seminorms $\mathcal{P}$. Since $\mathcal{P}$ is total and $x \neq x_0$, we have $p(x - x_0) > 0$ for some $p \in \mathcal{P}$. Then, $p \circ g(t_n) = t_n p(x - x_0) > 0$ for every $n \in \mathcal{N}$ and $\{p \circ g(t_n) \mid n \in \mathcal{N}\}$ is unbounded above. As p is continuous and (t_n) is unbounded, $\{p^{-1}([0, t_n)) \mid n \in \mathcal{N}\}$ is an open cover of $X - \{x_0\}$. As $\mathcal{X}$ is a topological vector space, $X - \{x_0\}$ is compact. Therefore, $\{p^{-1}([0, t_n)) \mid n \in \mathcal{N}\}$ contains a finite subcover. So, $\{g(t_n) \mid n \in \mathcal{N}\} \subset X - \{x_0\} \subset p^{-1}([0, t_{n_0}))$ for some $n_0 \in \mathcal{N}$. This implies $p \circ g(t_n) < t_{n_0}$ for every $n \in \mathcal{N}$, which is a contradiction.

(B) Combine Lemmas 4.3 and 4.4. ∎

5 Applications

In Section 5.1, we shall apply both the abstract duality results to a player's decision problem in a game $\Gamma \equiv \{N, (C_j, v_j)_{j \in N}\}$ with N as the set of players, C_j as player j's strategy space, $C \equiv \prod_{j \in N} C_j$ as the space of strategy profiles, and $v_j : C \to \Re$ as player j's utility function. In Section 5.2 (resp., 5.3), we shall apply the first (resp., second) duality result to a player's decision problem in the mixed extension of Γ, namely $\Gamma(m) \equiv \{N, (\Delta(C_j), V_j)_{j \in N}\}$, wherein $\Delta(C_j)$ is the set of player j's mixed strategies and $V_j : \prod_{k \in N} \Delta(C_k) \to \Re$ is player j's expected utility function derived from the von Neumann-Morgenstern utility v_j *via* the formula $V_j(\mu) = \int_C (\prod_{k \in N} \mu_k)(dc)\, v_j(c)$ for $\mu = (\mu_k)_{k \in N} \in \prod_{k \in N} \Delta(C_k)$.

We shall formulate some applications to welfare economics in Section 5.4. Our duality results seem apt for such applications because their formalism is quite general and not tied to a game's mixed extension.

5.1 Dualities and the abstract game

Given $i \in N$ and $C_{-i} \equiv \prod_{j \in N \setminus \{i\}} C_j$, player i's decision problem in Γ is (C_i, C_{-i}, v_i). When applying the duality results to this problem, C_i and v_i will directly meet the required hypotheses, while C_{-i} will inherit the required properties from $(C_j)_{j \in N \setminus \{i\}}$.

Applying Theorem 2.2 to (C_i, C_{-i}, v_i), we have the following result.

Theorem 5.1 *Suppose* $\Gamma = \{N, (C_j, v_j)_{j \in N}\}$ *and* $i \in N$ *are such that*

(a) N *is finite,*

(b) C_i *is a convex subset of a topological vector space,*

(c) for every $j \in N \setminus \{i\}$, C_j is a nonempty, convex, and compact subset of a locally convex topological vector space,

(d) v_i is continuous,

(e) $v_i(x, ty+(1-t)y') = tv_i(x,y)+(1-t)v_i(x,y')$ for all $x \in C_i$, $y, y' \in C_{-i}$, and $t \in (0,1)$, and

(f) $v_i(., y)$ is quasi-concave for every $y \in C_{-i}$.

Then, $x_0 \in C_i$ is not strongly dominated if and only if it is a best reply.

Given somewhat stronger hypotheses and by applying Theorem 2.3 to (C_i, C_{-i}, v_i), we have the following result.

Theorem 5.2 *Suppose $\Gamma = \{N, (C_j, v_j)_{j \in N}\}$, $i \in N$, and $x_0 \in C_i$ are such that*

(a) N is finite,

(b) C_i is a convex, compact, and metrisable subset of a Hausdorff locally convex topological vector space,

(c) for every $j \in N \setminus \{i\}$, C_j is a convex, compact, and metrisable subset of a locally convex topological vector space, with $C_j^* \neq \emptyset$,

(d) v_i is continuous,

(e) $v_i(x, ty+(1-t)y') = tv_i(x,y)+(1-t)v_i(x,y')$ for all $x \in C_i$, $y, y' \in C_{-i}$, and $t \in (0,1)$,

(f) $v_i(tx + (1-t)x', y) = tv_i(x,y) + (1-t)v_i(x', y)$ for all $x, x' \in C_i$, $y \in C_{-i}$, and $t \in (0,1)$, and

(g) $v_i(x, .) \neq v_i(x_0, .)$ for every $x \in C_i \setminus \{x_0\}$.

(A) If $x_0 \in \operatorname{Int} C_i$, then x_0 is weakly dominated if and only if it is not an internal-best reply.

(B) If $x_0 \in \operatorname{Fr} C_i$ and there is a compact set $C_i' \subset C_i \setminus \{x_0\}$ such that, for every $x \in C_i \setminus \{x_0\}$, $x_0 + t(x - x_0) \in C_i'$ for some $t \geq 1$, then x_0 is weakly dominated if and only if it is not an internal-best reply.

5.2 First duality and a game's mixed extension

Unlike in the previous section, we now consider a game Γ wherein the players' strategy spaces need not be convex subsets of linear spaces and their utilities need not have any geometric properties. As usual, this paucity of structure will be compensated for by taking recourse to Γ's mixed extension $\Gamma(m)$, which entails the interpretation of C_j as the set of player j's pure strategies and $v_j : C \to \Re$ as player j's von Neumann-Morgenstern utility. Player i's decision problem in $\Gamma(m)$ is specified as follows.

Definition 5.3 *Consider* $\Gamma = \{N, (C_j, v_j)_{j\in N}\}$ *and* $i \in N$ *such that*

(a) N *is finite,*

(b) for every $j \in N$, C_j *is a nonempty, compact, metric space, and*

(c) $v_i = \sum_{j\in N\setminus\{i\}} w_j \circ \pi_{i,j}$, *where* $\pi_{i,j}$ *projects* C *on* $C_i \times C_j$ *and* $w_j : C_i \times C_j \to \Re$ *is continuous for every* $j \in N \setminus \{i\}$.

Given Γ, *player* i*'s decision problem in its mixed extension* $\Gamma(m)$ *is* (X, Y, u) *wherein* $X \equiv \Delta(C_i)$, $Y \equiv \prod_{j\in N\setminus\{i\}} Y_j$ *with* $Y_j \equiv \Delta(C_j)$ *for every* $j \in N\setminus\{i\}$, *and* $u : X\times Y \to \Re$ *is given by* $u(x, y) = \int_C x\times\bar{y}(dc)\, v_i(c)$ *for* $(x, y) \in X\times Y$, *where* $\bar{y} \equiv \prod_{j\in N\setminus\{i\}} y_j$.

The function u is clearly identical to V_i. Hypothesis (c) requires comment. For every $j \in N \setminus \{i\}$, as $\pi_{i,j}$ and w_j are continuous, so is $w_j \circ \pi_{i,j}$. Therefore, as N is finite, v_i is continuous. If $|N| = 2$, then $\pi_{i,j}$ is the identity mapping on C and the assumed v_i amounts to no more than the specification of a continuous real-valued function on C. In this case, u clearly satisfies hypothesis (d) of Theorems 2.2 and 2.3. However, if $|N| > 2$, then u generated by an arbitrary continuous function $v_i : C \to \Re$ may not satisfy hypothesis (d) of Theorems 2.2 and 2.3 because the mapping $(y_j)_{j\in N\setminus\{i\}} \mapsto \prod_{j\in N\setminus\{i\}} y_j$ from profiles of measures in $\prod_{j\in N\setminus\{i\}} \Delta(C_j)$ to their Cartesian products in $\Delta(\prod_{j\in N\setminus\{i\}} C_j)$ is not affine in this case. Hypothesis (c) resolves this problem. Also note that Y is defined as $\prod_{j\in N\setminus\{i\}} \Delta(C_j)$ and not as $\Delta(C_{-i})$, i.e., player i's opponents are required to randomise independently.

Theorem 5.4 *If* (X, Y, u) *is given by Definition 5.3, then* $x_0 \in X$ *is not strongly dominated if and only if it is a best reply.*

Proof. We verify that (X, Y, u) satisfies Theorem 2.2's hypotheses. Then, the result follows from Theorem 2.2.

1. As *per* the discussion in Section 3, $\mathrm{rca}(C_j)$ with its weak* topology is a Hausdorff locally convex topological vector space for every $j \in N$. Therefore, $\prod_{j\in N\setminus\{i\}} \mathrm{rca}(C_j)$ is a Hausdorff locally convex topological

vector space. Moreover, $\Delta(C_j)$ is a convex, compact, and metrisable subset of $\mathrm{rca}(C_j)$ for every $j \in N$. Consequently, X and Y are convex, compact, and metrisable subsets of $\mathrm{rca}(C_i)$ and $\prod_{j\in N\setminus\{i\}} \mathrm{rca}(C_j)$ respectively. So, X and Y satisfy hypotheses (a) and (b).

2. Consider a sequence $(x^n, y^n) \subset X \times Y$ converging to $(x, y) \in X \times Y$ in the product topology of $X \times Y$. Then, $(x^n) \subset X$ weak* converges to $x \in X$ and $(y_j^n) \subset Y_j$ weak* converges to $y_j \in Y_j$ for every $j \in N \setminus \{i\}$. As each C_j is compact metric, so is C. Therefore, C is separable. Using Theorem 2.8 in Billingsley [2], the sequence $(x^n \times \bar{y}^n) \subset \Delta(C)$ weak* converges to $x \times \bar{y} \in \Delta(C)$. As v_i is continuous and bounded on C, $u(x^n, y^n) = \int_C x^n \times \bar{y}^n(dc)\, v_i(c) \to \int_C x \times \bar{y}(dc)\, v_i(c) = u(x, y)$. Since $X \times Y$ is metrisable, u is continuous and satisfies hypothesis (c).

3. Consider $(x, y) \in X \times Y$. Then, $u(x, y) = \int_C x \times \bar{y}(dc) \sum_{j\in N\setminus\{i\}} w_j \circ \pi_{i,j}(c) = \sum_{j\in N\setminus\{i\}} \int_C x \times \bar{y}(dc)\, w_j \circ \pi_{i,j}(c)$. Changing variables, we have $\int_C x \times \bar{y}(dc)\, w_j \circ \pi_{i,j}(c) = \int_{C_i\times C_j} (x \times \bar{y}) \circ \pi_{i,j}^{-1}(dc_i, dc_j)\, w_j(c_i, c_j) = \int_{C_i\times C_j} x \times y_j(dc_i, dc_j)\, w_j(c_i, c_j)$ for every $j \in N \setminus \{i\}$. So, $u(x, y) = \sum_{j\in N\setminus\{i\}} \int_{C_i\times C_j} x \times y_j(dc_i, dc_j)\, w_j(c_i, c_j)$. Using this formula, if $y, z \in Y$ and $t \in (0, 1)$, then it can be readily checked that $u(x, ty+(1-t)z) = tu(x, y) + (1-t)u(x, z)$. So, hypothesis (d) is satisfied.

4. Analogously, $u(tx + (1-t)x', y) = tu(x, y) + (1-t)u(x', y)$ for all $x, x' \in X$, $y \in Y$, and $t \in (0, 1)$. So, hypothesis (e) is satisfied. ∎

This result clearly applies in the special case wherein (X, Y, u) is generated by a finite game Γ.

5.3 Second duality and a game's mixed extension

Consider (X, Y, u) as *per* Definition 5.3. In order to apply Theorem 2.3 to this problem, we shall verify that its hypotheses are satisfied. The plan for doing this is as follows:

1. We shall re-formulate (X, Y, u) as an equivalent problem (K, L, w) so that various standard results, which are stated in the setting of vector spaces instead of affine spaces, can be applied directly to it. Lemma 5.6 will facilitate the embedding of L in an ambient space that meets the requirements of Theorem 2.3 and ensures that $L^* = \mathrm{Int}\, L \neq \emptyset$. This will enable the application of Theorem 2.3(A) to (K, L, w) and $x_0 \in \mathrm{Int}\, K$ in order to deduce Theorem 5.7.

2. Given the various properties of (K, L, w) verified in the proof of Theorem 5.7, the application of Theorem 2.3(B) to (K, L, w) and $x_0 \in \mathrm{Fr}\, K$

requires only that Theorem 2.3(B)'s supplementary hypothesis is satisfied. In the general case specified by Definition 5.3, we can show this for points in $\mathrm{Fr}\, K$ that satisfy a relative interiority condition. This condition is not vacuous as Lemma 5.9 shows that there are points in $\mathrm{Fr}\, K$ that do meet the condition. For such $x_0 \in \mathrm{Fr}\, K$, we can apply Theorem 2.3(B) and deduce Theorem 5.10. However, there remain points in $\mathrm{Fr}\, K$ that may not meet the relative interiority condition, e.g., the extreme points of K. Extendability of the weak dominance duality result to such points is an unresolved issue that may require a different technical approach.

3. However, if (K, L, w) is generated by a finite game, then it is readily shown in Theorem 5.11 that the supplementary hypothesis is satisfied for every $x_0 \in \mathrm{Fr}\, K$, thereby allowing the application of Theorem 2.3 to every $x_0 \in K$.

The first step of the plan is the re-formulation of player i's problem.

Definition 5.5 *Given (X, Y, u) and $i \in N$ as per Definition 5.3, $\alpha \in X$, and $\beta \equiv (\beta_j)_{j \in N \setminus \{i\}} \in Y$, player i's modified problem in $\Gamma(m)$ is (K, L, w), where $K \equiv X - \{\alpha\}$, $L \equiv \prod_{j \in N \setminus \{i\}} L_j$ with $L_j \equiv Y_j - \{\beta_j\}$, and $w : K \times L \to \Re$ is given by $w(x, y) = u(x + \alpha, y + \beta)$.*

It is easy to verify that $x \in K$ is strongly (resp., weakly) dominated with respect to (K, L, w) if and only if $x + \alpha \in X$ is strongly (resp., weakly) dominated with respect to (X, Y, u). Similarly, $x \in K$ is a best (resp., internal-best) reply with respect to (K, L, w) if and only if $x + \alpha \in X$ is a best (resp., internal-best) reply with respect to (X, Y, u). Since (K, L, w) and (X, Y, u) are equivalent insofar as the concepts defined in Definition 2.1 are concerned, we henceforth confine attention to (K, L, w).

As motivation for the following lemma, consider a set T with $|T| = n \in \mathcal{N}$. Given $\alpha \in \Delta(T)$, it is evident that $\mathcal{S} = \{x \in \Re^n \mid \sum_{i=1}^n x_i = 1\} - \{\alpha\}$ is a proper linear subspace of $\Re^n$, $\mathcal{S}$ is the linear span of $\Delta(T) - \{\alpha\}$, and $\mathrm{Int}\,[\Delta(T) - \{\alpha\}] \neq \emptyset$ relative to $\mathcal{S}$. These observations generalise as follows.

Lemma 5.6 *Suppose T is a compact metric space, $\alpha \in \Delta(T)$, and $\mathcal{S}$ is the closed linear span of $\Delta(T) - \{\alpha\}$ in $\mathrm{rca}(T)$. Then,*

(A) $\mathcal{S}$ is a Hausdorff locally convex topological vector space, and

(B) $\mathrm{Int}\,[\Delta(T) - \{\alpha\}] \neq \emptyset$ relative to $\mathcal{S}$.

The following result uses this lemma to derive the duality between the weak dominance property and the internal-best reply property for $x_0 \in \mathrm{Int}\, K$.

Theorem 5.7 *Suppose (K, L, w) is given by Definition 5.5 and $x_0 \in \operatorname{Int} K$ relative to the closed linear span of K. If $w(x_0, .) \neq w(x, .)$ for every $x \in K \setminus \{x_0\}$, then x_0 is weakly dominated if and only if it is not an internal-best reply.*

Proof. The result follows from Theorem 2.3(A) if (K, L, w) satisfies the hypotheses of Theorem 2.3.

Let $\mathcal{K}$ (resp., $\mathcal{L}_j$) be the closed linear span of $K = X - \{\alpha\} = \Delta(C_i) - \{\alpha\}$ (resp., $L_j = Y_j - \{\beta_j\} = \Delta(C_j) - \{\beta_j\}$). Using Lemma 5.6, $\mathcal{K}$ (resp., $\mathcal{L}_j$) is a Hausdorff locally convex topological vector space. Therefore, so is $\mathcal{L} = \prod_{j \in N \setminus \{i\}} \mathcal{L}_j$. Using the properties of X and Y derived in the proof of Theorem 5.4, K (resp., L) is a convex, compact, and metrisable subset of $\mathcal{K}$ (resp., $\mathcal{L}$). By Lemma 5.6, $\operatorname{Int} L_j \neq \emptyset$ relative to $\mathcal{L}_j$. As N is finite, $\operatorname{Int} L \neq \emptyset$ relative to $\mathcal{L}$. Therefore, $L^* = \operatorname{Int} L \neq \emptyset$ (Dunford and Schwartz [6], Theorem V.2.1). Consequently, K and L satisfy hypotheses (a) and (b).

Hypothesis (c) is satisfied by w because the product topology on $K \times L$ makes projections continuous, translations in a topological vector space are continuous, and the proof of Theorem 5.4 shows that u is continuous. Using the proof of Theorem 5.4, we can confirm that w satisfies hypotheses (d) and (e). ■

Finally, given (K, L, w), we use Theorem 2.3(B) to derive the duality between weak dominance and the internal-best reply property for $x_0 \in \operatorname{Fr} K$ that satisfies a relative interiority property. Given the proof of Theorem 5.7, the remaining requirement is the satisfaction of Theorem 2.3(B)'s supplementary hypothesis. The first step towards this verification is to decompose K and $\operatorname{Fr} K$ as *per* the formulae

$$K = \cap_{f \in \mathcal{F}} H_f^- \quad \text{and} \quad \operatorname{Fr} K = \cup_{f \in \mathcal{F}} (K \cap H_f) = \cup_{f \in \mathcal{F}} (K \cap A_f) \tag{1}$$

which we derive and interpret as follows.

1. Let $\mathcal{K}$ be the closed linear span of K. By Lemma 5.6, $\mathcal{K}$ is a Hausdorff locally convex topological vector space. Let $\mathcal{F}$ be the family of continuous non-zero linear functionals $f : \mathcal{K} \to \Re$. Each $f \in \mathcal{F}$ generates a supporting hyperplane of K, given by $H_f \equiv \{x \in \mathcal{K} \mid f(x) = \max f(K)\}$.[6] Then, $K = \cap_{f \in \mathcal{F}} H_f^-$, where $H_f^- \equiv \{x \in \mathcal{K} \mid f(x) \leq \max f(K)\}$ (Schaefer [11], Theorem II.10.1) is a closed supporting halfspace.

[6]A supporting hyperplane for K is a maximal proper affine subspace of $\mathcal{K}$ (i.e., with codimension 1) that is closed in $\mathcal{K}$, intersects K, and K is contained in one of the closed halfspaces generated by it (Schaefer [11], Sections I.4 and II.9). The other supporting hyperplane associated with f is $H_{-f} = \{x \in \mathcal{K} \mid -f(x) = \max(-f)(K)\} = \{x \in \mathcal{K} \mid f(x) = \min f(K)\}$.

2. Since $\text{Int}\, K \neq \emptyset$ relative to $\mathcal{K}$ by Lemma 5.6, K is a convex body (Schaefer [11], p. 40). It follows that $\text{Fr}\, K \subset \cup_{f\in\mathcal{F}} H_f$ (Schaefer [11], Corollary II.9.1). As K is closed in $\mathcal{K}$, we have $\text{Fr}\, K \subset K \cap (\cup_{f\in\mathcal{F}} H_f) = \cup_{f\in\mathcal{F}}(K \cap H_f) \subset \text{Fr}\, K$ (Schaefer [11], Lemma II.9.1). So, $\text{Fr}\, K = \cup_{f\in\mathcal{F}}(K \cap H_f)$ wherein $K \cap H_f$ is the facet of K generated by f.

3. Let A_f be the closed affine span of $K \cap H_f$. Evidently, $K \cap H_f \subset K \cap A_f$. As H_f is a closed affine subspace containing $K \cap H_f$, we have $A_f \subset H_f$. Therefore, $K \cap H_f \supset K \cap A_f$. Consequently, $\text{Fr}\, K = \cup_{f\in\mathcal{F}}(K \cap A_f)$.

The following example provides concrete manifestations of the abstract concepts used above. Specifically, it illustrates the difference between A_f and H_f, and also between the interior of facet $K \cap H_f$ relative to H_f and the interior of $K \cap A_f$ relative to A_f. Moreover, it suggests the source of these differences. It also prefigures the abstract facts derived in Lemma 5.9, and rationalises the assumptions made in these results.

Example 5.8 *Consider C_i with $|C_i| = 3$. Then, $\Delta(C_i)$ is the convex hull of the extreme points $e^1 = (1,0,0)$, $e^2 = (0,1,0)$, and $e^3 = (0,0,1)$. Given $\alpha \equiv (1/3,1/3,1/3) \in \Delta(C_i)$, let $K \equiv \Delta(C_i) - \{\alpha\} = \text{co}\,\{e^1-\alpha, e^2-\alpha, e^3-\alpha\}$. $\Delta(C_i)$ and K are convex and compact. The linear span of K is a hyperplane in $\Re^3$, namely the 2-dimensional subspace $\mathcal{K} = \{p \in \Re^3 \mid \sum_{i=1}^3 p_i = 0\}$.*

Consider $x_0 \equiv e^1 - \alpha \in \text{Fr}\, K$, which is an extreme point of K. A supporting hyperplane of K at x_0, relative to $\mathcal{K}$, is the 1-dimensional affine subspace $H_0 \equiv \{x_0 + t(e^3 - e^2) \mid t \in \Re\}$; H_0 is but one of an infinite family of supporting hyperplanes of K at x_0. Clearly, $K \cap H_0 = \{x_0\}$ and the affine span of $K \cap H_0$ is the 0-dimensional affine space $A_0 = \{x_0\}$. Observe that $K \cap H_0 = K \cap A_0 = \{x_0\}$, $A_0 \subset H_0$, $A_0 \neq H_0$, and

$$\text{Int}\,(K \cap H_0) = \text{Int}\,(K \cap A_0) = \begin{cases} \emptyset, & \textit{relative to } H_0 \\ \{x_0\}, & \textit{relative to } A_0. \end{cases}$$

Now consider $x_1 \equiv (e^1 + e^3)/2 - \alpha \in \text{Fr}\, K$. The unique supporting hyperplane of K at x_1, relative to $\mathcal{K}$, is the 1-dimensional affine subspace $H_1 \equiv \{x_1 + t(e^3 - e^1) \mid t \in \Re\}$. Then, $K \cap H_1 = \{x_1 + t(e^3 - e^1) \mid t \in [-1/2, 1/2]\}$ is a convex and compact facet of K and the affine span of $K \cap H_1$ is $A_1 = H_1$. So, $K \cap H_1 = K \cap A_1$ and $x_1 \in \text{Int}\,(K \cap H_1) = \text{Int}\,(K \cap A_1) = \{x_1 + t(e^3 - e^1) \mid t \in (-1/2, 1/2)\}$ relative to $H_1 = A_1$.

Note that H_1 is also a supporting hyperplane of K at x_0, relative to $\mathcal{K}$. However, unlike x_1, $x_0 \notin \text{Int}\,(K \cap H_1) = \text{Int}\,(K \cap A_1)$ relative to $H_1 = A_1$.

Let $H_2 \equiv \{e^2 - \alpha + t(e^1 - e^2) \mid t \in \Re\}$ and $H_3 \equiv \{e^3 - \alpha + t(e^2 - e^3) \mid t \in \Re\}$. Like H_1, H_2 and H_3 are 1-dimensional supporting hyperplanes of K relative to $\mathcal{K}$, which generate the facets $K \cap H_2 = \{e^2 - \alpha + t(e^1 - e^2) \mid t \in [0,1]\}$ and $K \cap H_3 = \{e^3 - \alpha + t(e^2 - e^3) \mid t \in [0,1]\}$. Then, $\text{Fr}\, K = \cup_{i=1}^3 (K \cap H_i)$, i.e., the frontier of K relative to $\mathcal{K}$ is the union of convex and compact facets.

Moreover, $x_1 \in \mathrm{Int}\,(K \cap H_1) = \mathrm{Int}\,(K \cap A_1)$ *relative to* $\mathrm{Fr}\,K$*, i.e., there is a set* V *that is open in* $\mathcal{K}$ *such that* $x_1 \in V \cap \mathrm{Fr}\,K \subset K \cap H_1 = K \cap A_1$*.*

As suggested by this example, we show more generally that a facet of K has a nonempty interior relative to its closed affine span. Moreover, if a frontier point of K belongs to a facet's interior relative to a supporting hyperplane, then it is in the facet's interior relative to the frontier of K.

Lemma 5.9 *Suppose* K *is given by Definition 5.5 and* $f \in \mathcal{F}$*. Then,*

(A) $K \cap A_f$ *is convex, compact, metrisable, and* $\mathrm{Int}\,(K \cap A_f) \neq \emptyset$ *relative to* A_f*, and*

(B) if $x_0 \in \mathrm{Int}\,(K \cap A_f)$ *relative to* A_f *and* $H_f = A_f$*, then* $x_0 \in \mathrm{Int}\,(K \cap A_f)$ *relative to* $\mathrm{Fr}\,K$*.*

We now use this result to verify the supplementary hypothesis of Theorem 2.3(B) for points in $\mathrm{Fr}\,K$ that satisfy the requirements of Lemma 5.9(B).

Theorem 5.10 *If* (K, L, w) *is given by Definition 5.5,* $f \in \mathcal{F}$*,* $x_0 \in \mathrm{Int}\,(K \cap H_f)$ *relative to* H_f*,* $H_f = A_f$*, and* $w(x_0, .) \neq w(x, .)$ *for every* $x \in K \setminus \{x_0\}$*, then* x_0 *is weakly dominated if and only if it is not an internal-best reply.*

Proof. Given the assumptions, we have already verified in the proof of Theorem 5.7 that all the hypotheses of Theorem 2.3 are satisfied. As $x_0 \in \mathrm{Fr}\,K$, it only remains to verify that the supplementary hypothesis of Theorem 2.3(B) holds. Let $\mathcal{T}$ be the topology of $\mathcal{K}$. As $\mathcal{K}$ is a Hausdorff locally convex topological vector space by Lemma 5.6, $\mathcal{T}$ is generated by a total family of seminorms $\mathcal{P}$.

By Lemma 5.9, there exists $V \in \mathcal{T}$ such that $x_0 \in V \cap \mathrm{Fr}\,K \subset K \cap H_f$. It follows that $C \equiv \mathrm{Fr}\,K \setminus V$ is closed in $\mathrm{Fr}\,K$ and $x_0 \notin C$. As $\mathrm{Fr}\,K$ is compact, C is compact. Moreover, $\mathrm{Fr}\,K \setminus H_f \subset \mathrm{Fr}\,K \setminus V = C$. Let $x \in K \setminus \{x_0\}$ and $h(t) \equiv x_0 + t(x - x_0)$ for $t \geq 1$. As $\mathcal{K}$ is Hausdorff and $x_0 \neq x$, there is a seminorm $p \in \mathcal{P}$ such that $p(x_0 - x) > 0$. Since $p(x_0 - h(t)) = p(t(x_0 - x)) = tp(x_0 - x)$, it follows that $\{p(x_0 - y) \mid y \in h([1, \infty))\}$ is unbounded above. We verify that $h(t) \in C$ for some $t \geq 1$.

Suppose $x \notin H_f$. If $h(t) \in H_f$ for some $t \geq 1$, then $x = t^{-1}h(t) + (1 - t^{-1})x_0 \in H_f$, a contradiction. So, $h(t) \notin H_f$ for every $t \geq 1$. As p is continuous and K is compact, $\{p(x_0 - y) \mid y \in K\}$ is bounded. Since $\{p(x_0 - y) \mid y \in h([1, \infty))\}$ is unbounded above, $h(t_0) \notin K$ for some $t_0 > 1$. Suppose $h^{-1}(\mathrm{Fr}\,K) = \emptyset$. Then, $h(1) = x \in K \setminus \mathrm{Fr}\,K = \mathrm{Int}\,K$. Consequently, $1 \in h^{-1}(\mathrm{Int}\,K)$, $t_0 \in h^{-1}(\mathcal{K} \setminus K)$, $[1, \infty) = h^{-1}(\mathrm{Int}\,K) \cup h^{-1}(\mathcal{K} \setminus K)$, and $h^{-1}(\mathrm{Int}\,K) \cap h^{-1}(\mathcal{K} \setminus K) = \emptyset$, which contradicts the fact that $[1, \infty)$ is topologically connected. Thus, $h(t_1) \in \mathrm{Fr}\,K \setminus H_f \subset C$ for some $t_1 \in [1, \infty)$.

Suppose $x \in H_f$. Then, $x \in K \cap H_f \subset \operatorname{Fr} K$. As p is continuous and $K \cap H_f$ is compact, $\{p(x_0 - y) \mid y \in K \cap H_f\}$ is bounded. Since $\{p(x_0 - y) \mid y \in h([1, \infty))\}$ is unbounded above, $h(t_0) \notin K \cap H_f$ for some $t_0 > 1$. So, there exists $t_1 \in [1, \infty)$ such that $h(t_1) \in \operatorname{Fr}(K \cap H_f) \subset K \cap H_f \subset \operatorname{Fr} K$. Therefore, $h(t_1) \notin \operatorname{Int}(K \cap H_f)$. As $V \cap \operatorname{Fr} K \subset \operatorname{Int}(K \cap H_f)$, we have $h(t_1) \notin V \cap \operatorname{Fr} K$. Thus, $h(t_1) \in \operatorname{Fr} K \setminus V = C$, as required. ■

A stronger result holds if (K, L, w) is generated by a finite game. In this case, the formulae in Equation (1) can be written in terms of a finite subfamily of supporting hyperplanes. This fact implies the duality between weak dominance and the internal-best reply property for every $x_0 \in K$.

Theorem 5.11 *Consider $\Gamma \equiv \{N, (C_j, v_j)_{j \in N}\}$ and $i \in N$ such that N and v_i satisfy Definition 5.3(a) and (c) respectively, and C_j is nonempty and finite for every $j \in N$. If (K, L, w) is generated from Γ via Definitions 5.3 and 5.5, $x_0 \in K$, and $w(x_0, .) \neq w(x, .)$ for every $x \in K \setminus \{x_0\}$, then x_0 is weakly dominated if and only if it is not an internal-best reply.*

Proof. As the hypothesised C_j satisfies Definition 5.3(b), Theorem 5.7 immediately yields the result if $x_0 \in \operatorname{Int} K$. Henceforth, let $x_0 \in \operatorname{Fr} K$. The hypothesised (K, L, w) is a special case of the problem considered in Theorem 5.7. This theorem's proof verified that (K, L, w) satisfies hypotheses (a)-(e) of Theorem 2.3. So, the application of Theorem 2.3(B) now only requires the satisfaction of its supplementary hypothesis.

Let $|C_i| = n+1$ for $n \in \mathcal{N}$, $J \equiv \{1, \ldots, n+1\}$, $X \equiv \Delta(C_i) = \operatorname{co}\{e^j \mid j \in J\}$, $a \in X$, $a \gg 0$, $K \equiv X - \{a\}$, and $E \equiv \{e^j - a \mid j \in J\}$.[7] So, $K = \operatorname{co} E$.

Let $\mathcal{K}$ be the n-dimensional linear span of K. As $a \gg 0$, $0 \in \operatorname{Int} K$ relative to $\mathcal{K}$. For $j \in J$, $H_j \equiv \{\sum_{i \in J} p_i(e^i - a) \mid p_j = 0 \ \wedge \ \sum_{i \in J} p_i = 1\}$ is the $(n-1)$-dimensional affine span of $E \setminus \{e^j - a\}$, i.e., H_j is a supporting hyperplane of K. Define the linear functional $f_j : \mathcal{K} \to \Re$ by $f_j(.) = -\langle e^j, . \rangle$, where $\langle ., . \rangle$ is the Euclidean inner product on $\mathcal{K}$. Clearly, $H_j = \{x \in \mathcal{K} \mid f_j(x) = a_j\}$ and $a_j = \max f_j(K)$.[8] Then, $\operatorname{Fr} K = \cup_{j \in J}(K \cap H_j)$ and $K = \cap_{j \in J} H_j^-$, where $H_j^- \equiv \{x \in \mathcal{K} \mid f_j(x) \leq a_j\}$.[9,10]

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[7] Example 5.8 sets $n = 2$. Here, e^j is the j-th unit vector of the canonical basis of $\Re^{n+1}$.

[8] The first claim follows as $E \setminus \{e^j - a\}$ contains n linearly independent vectors that are contained in the $(n-1)$-dimensional affine space $\{x \in \mathcal{K} \mid f_j(x) = a_j\}$. For the second claim, note that, if $x \in K$, then $x = \sum_{i \in J} p_i(e^i - a)$, where each $p_i \geq 0$ and $\sum_{i \in J} p_i = 1$. Then, $f_j(x) = a_j - p_j \leq a_j$ and $a_j = f_j(e^i - a) \in f_j(K)$ for $i \neq j$. So, $a_j = \max f_j(K)$.

[9] Clearly, $\operatorname{Fr} K \supset \cup_{j \in J}(K \cap H_j)$. Conversely, let $x \in \operatorname{Fr} K \subset K$. Then, there exists $p \in \Re^J_+$ such that $\sum_{i \in J} p_i = 1$ and $x = \sum_{i \in J} p_i(e^i - a)$. If $p_i > 0$ for every $i \in J$, then $x \in \operatorname{Int} K$, which contradicts $x \in \operatorname{Fr} K$. So, $p_j = 0$ for some $j \in J$. Therefore, $x \in K \cap H_j$.

[10] If $x \in K$, then $f_j(x) \leq \max f_j(K) = a_j$ for every j. So, $K \subset \cap_{j \in J} H_j^-$. Conversely, suppose $x \in \mathcal{K} \setminus K$. As $0 \in \operatorname{Int} K$, there exists $t \in (0, 1)$ such that $tx \in \operatorname{Fr} K = \cup_{j \in J}(K \cap H_j)$, i.e., $tx \in K \cap H_k$ for some k. So, $t f_k(x) = f_k(tx) = a_k$. Therefore, $f_k(x) = a_k / t > a_k$. It follows that $x \notin H_k^-$, and therefore, $x \notin \cap_{j \in J} H_j^-$.

Given $J_0 \equiv \{j \in J \mid x_0 \in H_j\}$, we have $n+1 > |J_0| > 0$. As each facet $K \cap H_j$ is compact and $|J \setminus J_0| \in \mathcal{N}$, it follows that $C \equiv \cup_{j \in J \setminus J_0}(K \cap H_j)$ is a nonempty and compact subset of $K \setminus \{x_0\}$.

Consider $x \in K \setminus \{x_0\}$. Define $h : [1, \infty) \to \mathcal{K}$ by $h(t) \equiv x_0 + t(x - x_0)$ for $t \geq 1$. As K is compact, it is closed in $\mathcal{K}$. Since h is continuous, $h^{-1}(K)$ is closed in $[1, \infty)$. Clearly, $1 \in h^{-1}(K)$. If $h^{-1}(K)$ is not bounded, then there is an unbounded, increasing sequence $(t_n) \subset h^{-1}(K)$. Then, $h(t_n) = x_0 + t_n(x - x_0) \in K$, i.e., $t_n(x - x_0) \in K - \{x_0\}$ for every $n \in \mathcal{N}$. As the norm is continuous, $\{\|y\| \mid y \in K - \{x_0\}\}$ is compact and therefore bounded. However, $t_n\|x - x_0\| = \|t_n(x - x_0)\| \in \{\|y\| \mid y \in K - \{x_0\}\}$ for every $n \in \mathcal{N}$. As $x \neq x_0$, $\|x - x_0\| > 0$ and $(t_n\|x - x_0\|)$ is unbounded, which is a contradiction. So, $h^{-1}(K)$ is bounded. Consequently, $h^{-1}(K)$ is nonempty and compact. It follows that there exists $t_0 \equiv \max h^{-1}(K)$, i.e., $t_0 \geq 1$ such that $h(t_0) \in K$ and $h(t) \notin K$ for $t > t_0$. Clearly, $h(t_0) \in \operatorname{Fr} K$.

With this preparation, we now show that, for $x \in K \setminus \{x_0\}$, there exists $t \in [1, \infty)$ such that $h(t) \in C$.

1. Suppose $x \in \operatorname{Int} K$ relative to $\mathcal{K}$. Then, $t_0 \equiv \max h^{-1}(K) > 1$. Suppose $h(t_0) \notin C$. Then, $h(t_0) \in \cup_{j \in J_0}(K \cap H_j)$. If $h(t_0) \in K \cap H_j$ for $j \in J_0$, then the convexity of $K \cap H_j$ implies $x = t_0^{-1}h(t_0) + (1 - t_0^{-1})x_0 \in K \cap H_j \subset \operatorname{Fr} K$, which contradicts $x \in \operatorname{Int} K$. So, $h(t_0) \in C$.

2. Suppose $x \in \cup_{j \in J \setminus J_0}(K \cap H_j)$. Then, $h(1) = x \in C$.

3. Finally, suppose $x \in \cup_{j \in J_0}(K \cap H_j)$ and $h(t_0) \notin C$. We derive a contradiction. Let $J_x \equiv \{j \in J_0 \mid x \in H_j\}$. It follows that $h([1, \infty)) \subset \cap_{j \in J_x} H_j$. Note the inclusions $J_x \subset J_0 \subset J$.

   (a) Consider $j \in J \setminus J_0$. As $h(t_0) \in K \setminus C$, we have $f_j(h(t_0)) < a_j$.

   (b) Consider $j \in J_0 \setminus J_x$. As $f_j(x_0) = a_j > f_j(x)$, we have $f_j(h(t_0)) = f_j(x_0) + t_0[f_j(x) - f_j(x_0)] < f_j(x_0) = a_j$.

   (c) Consider $j \in J_x$. Then, $f_j(x_0) = f_j(x) = a_j = f_j(h(t))$ for every $t \geq 1$.

   (d) By steps (a), (b), and the finiteness of $J \setminus J_x$, there exists $\epsilon > 0$ such that $f_j(h(t_0 + \epsilon)) < a_j$ for every $j \in J \setminus J_x$. By step (c), $f_j(h(t_0 + \epsilon)) = a_j$ for every $j \in J_x$. It follows that $h(t_0 + \epsilon) \in \cap_{j \in J} H_j^- = K$, which contradicts the definition of t_0. ■

5.4 Characterisations of efficiency

This section concerns applications of our results to decision problems that are not game-theoretic in nature and do not involve probabilistic notions such as a belief about the states or a randomisation over actions. Although the probabilistic manifestations of the abstract dualities, which are applied in Sections 5.2 and 5.3, are inapplicable to these problems, the abstract duality results in Section 2 are applicable to them.

Consider a set of agents N, a parameter space T, and a planner who chooses an outcome (or in terms of our formalism, an action) from the set X. Given $v : X \times T \times N \to \Re$, agent i's utility function is $v(.,.,i)$.

Definition 5.12 *An outcome $x_0 \in X$ is said to be weakly (resp., strongly) Pareto efficient if it is not strongly (resp., weakly) dominated with respect to the problem $(X, T \times N, v)$, i.e., there is no $x \in X$ such that $v(x_0,.,.) < v(x,.,.)$ (resp., $v(x_0,.,.) \leq v(x,.,.)$ and $v(x_0,.,.) \neq v(x,.,.)$).*

This notion does not aggregate across agents. It is straightforward to make assumptions on the problem $(X, T \times N, v)$ that meet the requirements of Theorems 2.2 and 2.3. Clearly, v has to be continuous. Among other hypotheses, N has to be infinite, say $N = [0, 1]$, in order to satisfy hypothesis (b) of Theorems 2.2 and 2.3. Given appropriate assumptions, $x_0 \in X$ is weakly (resp., strongly) Pareto efficient if and only if it is a best (resp., internal-best) reply with respect to $(X, T \times N, v)$.

Now define $\hat{v} : X \times T \to \Re^N$ by $\hat{v}(x,t)(.) = v(x,t,.)$. Let $F : \Re^N \to \Re$ be an increasing function, i.e., $x \geq y$ and $x \neq y$ implies $F(x) > F(y)$, where the relations $\geq$ and $=$ on $\Re^N$ are defined pointwise. Suppose the planner's welfare function is $w : X \times T \to \Re$, given by $w = F \circ \hat{v}$, i.e., F aggregates the profile of agents' utilities $\hat{v}(x,t)$ to yield the planner's welfare $w(x,t)$.

Definition 5.13 *An outcome $x_0 \in X$ is said to be weakly (resp., strongly) Utilitarian efficient if it is not strongly (resp., weakly) dominated with respect to the problem (X, T, w), i.e., there is no $x \in X$ such that $w(x_0,.) < w(x,.)$ (resp., $w(x_0,.) \leq w(x,.)$ and $w(x_0,.) \neq w(x,.)$).*

This notion aggregates across agents. If (X, T, w) satisfies the requirements of Theorem 2.2 (resp., 2.3), then x_0 is weakly (resp., strongly) Utilitarian efficient if and only if it is a best (resp., internal-best) reply with respect to (X, T, w).[11] Moreover, as F is increasing, an outcome's weak (resp., strong) Utilitarian efficiency implies its weak (resp., strong) Pareto efficiency.

The above definitions and results are easily specialised if the agents' utilities are given by $v : X \times N \to \Re$. Then, $\hat{v} : X \to \Re^N$ and $w : X \to \Re$ are given by $\hat{v}(x)(.) = v(x,.)$ and $w = F \circ \hat{v}$ respectively. Since T is irrelevant in this case, we may specify it as the trivial vector space.

[11] The properties of w can be inferred from those of v and F. Suppose X, T, and N are topological spaces, and $\Re^N$ is given the compact-open topology. If v and F are continuous, then so are $\hat{v}$ (Dugundji [5], Theorem XII.3.1) and w. Suppose F is affine. If $x \in X$ and $v(x,.,i)$ is affine for every $i \in N$, then $w(x,.)$ is affine. If $t \in T$ and $v(.,t,i)$ is concave (resp., affine) for every $i \in N$, then $w(.,t)$ is concave (resp., affine).

6 Concluding remarks

We have demonstrated dualities between actions that are not strongly (resp., weakly) dominated and actions that are best (resp., internal-best) replies in the setting of action and state spaces that are subsets of abstract topological vector spaces. These results are applied to a player's decision problem in a many-player game when the players' strategy spaces are subsets of abstract topological vector spaces. They are also applied to a player's decision problem in the mixed extension of a many-player game with compact metric spaces of pure strategies. Finally, the dualities are used to characterise various notions of efficient outcomes in terms of different kinds of best replies.

The problem of extending, if possible, the scope of Theorem 5.10 to all the frontier points of K remains open.

A Appendix

Lemma A.1 *Suppose* (X, Y, u) *is such that* X *and* Y *are nonempty compact Hausdorff spaces, and* u *is continuous, then there exists* $x_0 \in X$ *that is not weakly dominated.*

Proof. Let $\Re^Y$ be the space of continuous functions $f : Y \to \Re$. Give $\Re^Y$ the compact-open topology. As $\Re$ is Hausdorff, so is $\Re^Y$ (Dugundji [5], Theorem XII.1.3). The mapping $F : X \to \Re^Y$, defined by $F(x)(.) = u(x, .)$, is continuous (Dugundji [5], Theorem XII.3.1). Therefore, as X is compact, so is $F(X)$.

For $f, g \in F(X)$, we say $f \succ g$ if and only if $f(.) \geq g(.)$ on Y. Clearly, $\succ$ is a preordering on $F(X)$. Moreover, $x_0 \in X$ is not weakly dominated if and only if, for $g \in \Re^Y$, $g \succ F(x_0)$ and $g \in F(X)$ implies $g = F(x_0)$.

Let K_0 be a chain in $F(X)$. We show that K_0 has an upper bound in $F(X)$, i.e., there exists $g \in F(X)$ such that $g \succ f$ for every $f \in K_0$.

Consider $f \in K_0$. Then, $S(f) \equiv \{x \in X \mid F(x) \succ f\} \neq \emptyset$ as $f \in F(X)$. Clearly, $\{g \in \Re^Y \mid g \succ f\} = \Re^Y \setminus \cup_{y \in Y}\{g \in \Re^Y \mid g(y) < f(y)\}$. Given the compact-open topology of $\Re^Y$, $\{g \in \Re^Y \mid g(y) < f(y)\}$ is open in $\Re^Y$ for every $y \in Y$. Therefore, $\{g \in \Re^Y \mid g \succ f\}$ is closed in $\Re^Y$. As F is continuous, $S(f) = F^{-1}(\{g \in \Re^Y \mid g \succ f\})$ is closed in X. Since X is compact, $S(f)$ is compact. As F is continuous, $F \circ S(f)$ is compact and nonempty. Since $\Re^Y$ is Hausdorff, $F \circ S(f)$ is closed in $\Re^Y$.

Given $\{f_1, \ldots, f_n\} \subset K_0$, we have $\cap_{i=1}^n F \circ S(f_i) \neq \emptyset$ as K_0 is a chain. It follows that $\cap_{f \in K_0} F \circ S(f) \neq \emptyset$ (Dugundji [5], Theorem XI.1.3). Consider $g \in \cap_{f \in K_0} F \circ S(f)$ and $f \in K_0$. Then, $g \in F \circ S(f)$, i.e., $g = F(x)$ for some $x \in S(f)$. As $x \in S(f)$, $F(x) \succ f$. So, $g \succ f$. Thus, g is an upper bound of K_0. By Zorn's lemma (Dugundji [5], Theorem II.2.1), there exists a maximal element $g \in F(X)$.

As $g \in F(X)$, we have $g = F(x_0)$ for some $x_0 \in X$, i.e., $g(.) = u(x_0, .)$. Suppose x_0 is weakly dominated. Then, there exists $x \in X$ such that $F(x)(.) = u(x, .) \geq u(x_0, .) = g(.)$ and $F(x)(y) = u(x, y) > u(x_0, y) = g(y)$ for some $y \in Y$. So, $F(x) \succ g$ and $\neg g \succ F(x)$. But, as g is maximal, we have $g \succ F(x)$, a contradiction. ■

Proof of Lemma 3.2 Consider $x \in K$. As K is metrisable, it suffices to find a sequence in K^* converging to x.

Suppose $0_{\mathcal{K}} \in K^*$. If f is the support function of K, then $f(x) \in [0, 1]$. As K is convex, for $n \in \mathcal{N}$, we have $x_n = [(n-1)/n]x \in K$ and $f(x_n) = f([(n-1)/n]x) = [(n-1)/n]f(x) \in [0, 1)$, i.e., $x_n \in K^*$. So, the sequence $(x_n) \subset K^*$ converges to x.

Suppose $0_{\mathcal{K}} \notin K^*$. As $K^* \neq \emptyset$, there exists $x_0 \in K^*$. As $\mathcal{K}$ is a topological vector space, $L = K - \{x_0\}$ is a convex and metrisable subset of $\mathcal{K}$. Using the definition of internal points, $y \in K^*$ if and only if $y - x_0 \in L^*$. So, $0_{\mathcal{K}} \in L^*$. By step (1), L^* is dense in L. As $x - x_0 \in L$, there is a sequence $(y_n) \subset L^*$ converging to $x - x_0$. So, the sequence $(y_n + x_0) \subset K^*$ converges to x. ■

Proof of Lemma 3.4 We do the proof in two steps.

1. Suppose $0_{\mathcal{Y}} \in Y^*$. Define $\phi_n : \mathcal{Y} \to \mathcal{Y}$ by $\phi_n(y) = [(n-1)/n]y$ for $n \in \mathcal{N}$ and $y \in \mathcal{Y}$. Let $K_n = \phi_n(Y)$. Clearly, $0_{\mathcal{Y}} = \phi_n(0_{\mathcal{Y}}) \in \phi_n(Y) = K_n$ for every $n \in \mathcal{N}$. K_n is convex as Y is convex and ϕ_n is linear. K_n is compact as ϕ_n is continuous and Y is compact. So, K_n is nonempty, convex, and compact for every $n \in \mathcal{N}$.

   Consider $z \in K_n$ for $n \in \mathcal{N}$. Then, $z = [(n-1)/n]y$ for some $y \in Y$. It follows that $z = [n/(n+1)]y'$ where $y' = (1 - 1/n^2)y$. As Y is convex and $0_{\mathcal{Y}} \in Y$, we have $y' \in Y$. Therefore, $z = \phi_{n+1}(y') \in \phi_{n+1}(Y) = K_{n+1}$. So, $K_n \subset K_{n+1}$ for every $n \in \mathcal{N}$.

   Let f be the support function of Y. Consider $z \in K_n$ for some $n \in \mathcal{N}$. Then, $z = \phi_n(y) = [(n-1)/n]y$ for some $y \in Y$. As $y \in Y$, $f(y) \in [0, 1]$. Therefore, $f(z) = f([(n-1)/n]y) = [(n-1)/n]f(y) < 1$. So, $z \in Y^*$. Conversely, let $z \in Y^*$. So, there exists $\epsilon > 0$ such that $|\delta| < \epsilon$ implies $(1+\delta)z = z + \delta z \in Y$. Let $n \in \mathcal{N}$ be such that $n > 1$ and $1/(n-1) < \epsilon$. Then, $y \equiv [1 + 1/(n-1)]z \in Y$ and $z = [(n-1)/n]y = \phi_n(y) \in K_n$. So, $Y^* = \cup_{n \in \mathcal{N}} K_n$. Combining the above arguments, $\{K_n \mid n \in \mathcal{N}\}$ internally approximates Y.

2. Suppose $0_{\mathcal{Y}} \notin Y^*$. As $Y^* \neq \emptyset$, there exists $y_0 \in Y^*$. Define $T : \mathcal{Y} \to \mathcal{Y}$ by $T(y) = y - y_0$. As $\mathcal{Y}$ is a topological vector space, T is an affine homeomorphism. So, $T(Y)$ is a convex and compact subset of $\mathcal{Y}$, and $0_{\mathcal{Y}} \in T(Y)^*$. By step 1, there is a family of sets $\{K_n \mid n \in \mathcal{N}\}$ that internally approximates $T(Y)$. The function inverse of T is T^{-1}, given

by $T^{-1}(y) = y + y_0$. We verify that the family of sets $\{T^{-1}(K_n) \mid n \in \mathcal{N}\}$ internally approximates Y.

Consider $n \in \mathcal{N}$. $T^{-1}(K_n) \neq \emptyset$ as $K_n \neq \emptyset$. Since K_n is convex and T^{-1} is affine, $T^{-1}(K_n)$ is convex. As K_n is compact and T^{-1} is continuous, $T^{-1}(K_n)$ is compact. As $K_n \subset K_{n+1}$, we have $T^{-1}(K_n) \subset T^{-1}(K_{n+1})$. Finally, as $T(Y)^* = \cup_{n\in\mathcal{N}} K_n$, we have $T^{-1}(T(Y)^*) = \cup_{n\in\mathcal{N}} T^{-1}(K_n)$. So, it suffices to show that $T^{-1}(T(Y)^*) = Y^*$, or equivalently, $T(Y)^* = T(Y^*)$.

Consider $y \in T(Y^*)$ and $z \in \mathcal{Y}$. Then, $y = x - y_0$ for some $x \in Y^* \subset Y$. So, $y \in T(Y)$. As $x \in Y^*$, there exists $\epsilon > 0$ such that $x + \delta z \in Y$ for $|\delta| < \epsilon$, which implies $y + \delta z = x + \delta z - y_0 \in T(Y)$ for $|\delta| < \epsilon$. Thus, $y \in T(Y)^*$.

Conversely, consider $y \in T(Y)^* \subset T(Y)$ and $z \in \mathcal{Y}$. Then, $y = x - y_0$ for some $x \in Y$. As $y \in T(Y)^*$, there exists $\epsilon > 0$ such that $y + \delta z \in T(Y)$ for $|\delta| < \epsilon$, which implies $x + \delta z = y + \delta z + y_0 \in Y$ for $|\delta| < \epsilon$. Thus, $x \in Y^*$ and $y \in T(Y^*)$. ■

Proof of Lemma 5.6 We shall use the properties of $\text{rca}(T)$ and $\Delta(T)$ with respect to the weak* topology of $\text{rca}(T)$, as outlined in Section 3.

Given $\alpha \in \Delta(T)$, as $x \mapsto x - \alpha$ is an affine homeomorphism on $\text{rca}(T)$, $K \equiv \Delta(T) - \{\alpha\}$ is a convex, compact, metrisable, and separable subset of B_2, with $0 \in K$. As K is separable, there is a countable set K' that is dense in K.

Let $\mathcal{S}$ be the closed linear span of K. Given its subspace topology derived from $\text{rca}(T)$, $\mathcal{S}$ is a Hausdorff locally convex topological vector space. We show that $\text{Int}\, K \neq \emptyset$ relative to $\mathcal{S}$.

Let $\{h_i \mid i \in I\}$ be the family of non-zero continuous real-valued linear functionals on $\mathcal{S}$. Each h_i generates the closed halfspace $H_i^- \equiv \{x \in \mathcal{S} \mid h_i(x) \leq c_i\} \supset K$ with $c_i \equiv \max h_i(K)$. It follows that $K = \cap_{i\in I} H_i^-$ (Schaefer [11], Theorem II.10.1).

Consider $\mathcal{S} \cap B_2$ with the subspace topology. As B_2 is metrisable, so is $\mathcal{S} \cap B_2$. Let $H_i^+ = (\mathcal{S} \cap B_2) \setminus H_i^-$. Then, H_i^+ is open in $\mathcal{S} \cap B_2$. Since K is a compact subset of $\mathcal{S} \cap B_2$, it is closed in $\mathcal{S} \cap B_2$. Therefore, $\text{Fr}\, K = \overline{K} \cap \overline{(\mathcal{S} \cap B_2) \setminus K} = K \cap \overline{(\mathcal{S} \cap B_2) \setminus \cap_{i\in I} H_i^-} = K \cap \overline{\cup_{i\in I} H_i^+}$.

$\mathcal{H} = \{H_i^+ \mid i \in I\}$ is an open cover of $(\mathcal{S} \cap B_2) \setminus K$. As $(\mathcal{S} \cap B_2) \setminus K$ is metrisable, it is paracompact (Aliprantis and Border [1], Theorem 3.22). Therefore, $\mathcal{H}$ has a locally finite refinement consisting of open sets $\{A_j \mid j \in J\}$ such that $(\mathcal{S} \cap B_2) \setminus K \subset \cup_{j\in J} A_j$, and for every $j \in J$, there exists an $i \in I$ such that $A_j \subset H_i^+$. So, $\cup_{j\in J} A_j \subset \cup_{i\in I} H_i^+$. As $H_i^+ \subset (\mathcal{S} \cap B_2) \setminus K \subset \cup_{j\in J} A_j$ for every $i \in I$, we have $\cup_{j\in J} A_j \supset \cup_{i\in I} H_i^+$. Therefore, $\cup_{j\in J} A_j = \cup_{i\in I} H_i^+$. Since $\{A_j \mid j \in J\}$ is locally finite, we have

$\overline{\cup_{j\in J}A_j} = \cup_{j\in J}\overline{A_j}$.[12] Consequently, $\operatorname{Fr} K = K \cap \overline{\cup_{j\in J}A_j} = K \cap (\cup_{j\in J}\overline{A_j})$.

Consider $x \in \operatorname{Fr} K$. As $x \in \operatorname{Fr} K \subset K = \cap_{i\in I} H_i^-$, we have $h_i(x) \leq c_i$ for every $i \in I$. As $x \in \operatorname{Fr} K = K \cap (\cup_{j\in J}\overline{A_j})$, we have $x \in \overline{A_j}$ for some $j \in J$. Therefore, $x \in \overline{H_i^+}$ for some $i \in I$. For this i, $h_i(x) \geq c_i$, and so, $h_i(x) = c_i$.

Let $K' = \{x_n \mid n \in \mathcal{N}\}$ and $z_n = \sum_{k=1}^{n} 2^{-k} x_k$ for $n \in \mathcal{N}$. Then, $z_n \in K$ as $K' \subset K$, $0 \in K$, and K is convex. As K is a compact metric space, the sequence (z_n) has a cluster point $\bar{z} \in K$.

Suppose $\bar{z} \in \operatorname{Fr} K$. Therefore, $h_i(\bar{z}) = c_i$ for some $i \in I$. As $K' \subset K$, $h_i \leq c_i$ on K'. Suppose $h_i = c_i$ on K'. As K' is dense in K and h_i is continuous, $h_i = c_i$ on K. As $0 \in K$, we have $c_i = h_i(0) = 0$. So, $h_i = 0$ on K. As h_i is linear, $h_i = 0$ on the linear span of K. As h_i is continuous, $h_i = 0$ on $\mathcal{S}$, which contradicts the assumption that h_i is non-zero on $\mathcal{S}$. So, there exists $n \in \mathcal{N}$ and $\epsilon > 0$ such that $h_i(x_n) = c_i - \epsilon$. By definition, $\bar{z} = \lim_{m\uparrow\infty} z_{n_m} = \lim_{m\uparrow\infty} \sum_{k=1}^{n_m} 2^{-k} x_k$ for some strictly increasing sequence $(n_m) \subset \mathcal{N}$. As h_i is linear, for every $m \in \mathcal{N}$ such that $n_m > n$, we have $h_i(z_{n_m}) = h_i\left(\sum_{k=1}^{n_m} 2^{-k} x_k\right) = \sum_{k=1}^{n_m} 2^{-k} h_i(x_k) \leq \sum_{k=1}^{n_m} 2^{-k} c_i - 2^{-n}\epsilon$. As h_i is continuous, $c_i = h_i(\bar{z}) = \lim_{m\uparrow\infty} h_i(z_{n_m}) \leq c_i - 2^{-n}\epsilon < c_i$, which is a contradiction. Therefore, $\bar{z} \in \operatorname{Int} K$. ■

Proof of Lemma 5.9 Let K and f be as hypothesised.

(A) A_f is closed and convex. As the proof of Theorem 5.7 shows that K is convex, compact, and metrisable, it follows that $K \cap A_f$ also has these properties. We now show that $\operatorname{Int}(K \cap A_f) \neq \emptyset$ relative to A_f.

Let $D : C_i \to \Delta(C_i)$ map $c \in C_i$ to the Dirac measure $\delta_c \in \Delta(C_i)$. As D is an imbedding (Parthasarathy [9], Lemma II.6.1), $D(C_i)$ is a compact metric space. Evidently, δ_c is an extreme point of $\Delta(C_i)$ for every $c \in C_i$. Conversely, suppose z is an extreme point of $\Delta(C_i)$. If $z(A) \in (0,1)$ for some $A \in \mathcal{B}(C_i)$, then define $x, y \in \Delta(C_i)$ by $x(.) = z(. \cap A)/z(A)$ and $y(.) = z(. \setminus A)/[1 - z(A)]$. As $z = z(A)x + (1 - z(A))y$, z is not an extreme point of $\Delta(C_i)$, which is a contradiction. So, $z(A) \in \{0,1\}$ for every $A \in \mathcal{B}(C_i)$. Suppose $c, c' \in \operatorname{supp} z$ and $c \neq c'$. As C_i is Hausdorff, there exist disjoint open neighbourhoods U and V of c and c' respectively. As $c, c' \in \operatorname{supp} z$, we have $z(U) = 1 = z(V)$, which contradicts $z \in \Delta(C_i)$. So, $\operatorname{supp} z$ is a singleton. Therefore, $D(C_i)$ is the set of extreme points of $\Delta(C_i)$ and $E \equiv D(C_i) - \{\alpha\}$ is the compact metric space of extreme points of $K = \Delta(C_i) - \{\alpha\}$.

As $K \cap A_f = K \cap H_f$, and $E \subset K$, Theorem II.10.3 in Schaefer [11] implies $E_0 \equiv E \cap K \cap A_f = E \cap K \cap H_f = E \cap H_f \neq \emptyset$. Clearly, E_0 is a compact metric

[12] For every $k \in J$, as $A_k \subset \cup_{j\in J} A_j$, we have $\overline{A_k} \subset \overline{\cup_{j\in J} A_j}$. So, $\cup_{j\in J}\overline{A_j} \subset \overline{\cup_{j\in J} A_j}$. For the converse, it is sufficient to show that $\cup_{j\in J}\overline{A_j}$ is a closed set. Consider $x \notin \cup_{j\in J}\overline{A_j}$. By local finiteness, there exists an open neighbourhood U of x and a finite set $J' \subset J$ such that $U \cap A_j = \emptyset$ for every $j \in J \setminus J'$. So, $V = U \setminus (\cup_{j\in J'}\overline{A_j})$ is an open neighbourhood of x and $V \cap A_j = \emptyset$ for every $j \in J$. Therefore, $V \cap \overline{A_j} = \emptyset$ for every $j \in J$, i.e., $V \cap (\cup_{j\in J}\overline{A_j}) = \emptyset$. So, the complement of $\cup_{j\in J}\overline{A_j}$ is open and $\cup_{j\in J}\overline{A_j}$ is closed.

space and every $x \in E_0$ is an extreme point of $K \cap A_f$. Conversely, suppose $x \in (K \cap A_f) \setminus E$. Then, $f(x) = \max f(K)$ and $x = ty + (1-t)z$ for some $t \in (0, 1)$ and $y, z \in K$. It follows that $\max f(K) = f(x) = tf(y) + (1-t)f(z)$. If $f(y) < \max f(K)$, then $f(z) > \max f(K)$, which contradicts $z \in K$. So, $f(y) = f(z) = \max f(K)$, i.e., $y, z \in K \cap H_f = K \cap A_f$. Therefore, x is not an extreme point of $K \cap A_f$. So, an extreme point of $K \cap A_f$ must belong to E. Thus, E_0 is the set of extreme points of $K \cap A_f$.

Let F be the translation $x \mapsto x + \alpha$ on $\text{rca}(C_i)$. Then, $F(K \cap A_f)$ is a convex, compact, and metrisable subset of $\Delta(C_i)$, with $F(E_0)$ as the set of extreme points. By the Krein-Milman theorem (Schaefer [11], Theorem II.10.4), $F(K \cap A_f) = \overline{\text{co}}\, F(E_0)$.

By Lemma V.2.4 in Dunford and Schwartz [6], $\overline{\text{co}}\, F(E_0) = \overline{\text{co}\, F(E_0)}$. As $E_0 \subset E$, we have $F(E_0) \subset F(E) = E + \{\alpha\} = D(C_i)$. So, $C_i^0 \equiv D^{-1} \circ F(E_0) \subset C_i$. As D^{-1} and F are continuous, C_i^0 is a compact metric subset of C_i. So, $\Delta(C_i^0)$ is a compact metric space (Parthasarathy [9], Theorem 6.4). It follows that $\overline{\text{co}\, F(E_0)} = \overline{\text{co}\, D(C_i^0)} \subset \Delta(C_i^0)$. As C_i^0 is separable metric, $\overline{\text{co}\, D(C_i^0)} \supset \Delta(C_i^0)$ (Parthasarathy [9], Theorem II.6.3). We conclude that $F(K \cap A_f) = \Delta(C_i^0)$, i.e., $K \cap A_f = \Delta(C_i^0) - \{\alpha\}$.

Let $\gamma \in \Delta(C_i^0)$. As A_f is the closed affine span of $K \cap H_f = K \cap A_f$, $A_f + \{\alpha - \gamma\}$ is the closed linear span of $(K \cap A_f) + \{\alpha - \gamma\} = \Delta(C_i^0) - \{\gamma\}$. By Lemma 5.6, $\text{Int}\,[\Delta(C_i^0) - \{\gamma\}] \neq \emptyset$ relative to $A_f + \{\alpha - \gamma\}$. So, $\text{Int}\,(K \cap A_f) = \text{Int}\,[\Delta(C_i^0) - \{\alpha\}] \neq \emptyset$ relative to A_f.

(B) Suppose the hypotheses are satisfied and x_0 is not an interior point of $K \cap A_f$ relative to $\text{Fr}\, K$. As $A_f = H_f$, there is a sequence $(x_n) \subset \text{Fr}\, K \setminus (K \cap H_f) = \text{Fr}\, K \setminus H_f$ converging to x_0.

Let $H_f = \{x \in \mathcal{K} \mid f(x) = \max f(K)\}$. As $x_0 \in H_f$ and $x_n \in K \setminus H_f$, $f(x_0) > f(x_n)$ for every $n \in \mathcal{N}$. By Lemma 5.6, there exists $c \in \text{Int}\, K$. Since $\alpha \equiv x_0 - c \neq 0$ and $c \notin H_f$, we have $f(\alpha) = f(x_0) - f(c) > 0$. Setting $t_n \equiv [f(x_0) - f(x_n)]/f(\alpha) > 0$ and $y_n \equiv x_n + t_n \alpha$, we have $f(y_n) = f(x_n) + t_n f(\alpha) = f(x_0)$. So, $y_n \in H_f$. Set $z_n \equiv y_n - \alpha$. As $\lim_n x_n = x_0$, we have $\lim_n t_n = 0$. Therefore, $\lim_n y_n = \lim_n x_n = x_0$ and $\lim_n z_n = c$.

Let U be an open neighbourhood of c such that $U \subset K$. As $\lim_n x_n = x_0$, we may assume that $(x_n) \subset U + \{\alpha\}$ and $f(x_n) > f(c)$ for every $n \in \mathcal{N}$. Then, $t_n < 1$ for every $n \in \mathcal{N}$. As $\lim_n y_n = x_0$, there exists $N \in \mathcal{N}$ such that $n > N$ implies $y_n \in U + \{\alpha\}$ and $z_n \in U$. As $z_n \in \text{Int}\, K$, $x_n \in \text{Fr}\, K$, and $x_n = (1 - t_n) y_n + t_n z_n$ for $n > N$, we have $y_n \notin K$; otherwise, $x_n \in \text{Int}\, K$, which is a contradiction. Therefore, $y_n \in H_f \setminus K \subset H_f \setminus (K \cap H_f)$ for $n > N$, which contradicts $x_0 \in \text{Int}\,(K \cap H_f)$ relative to H_f. ■

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