Externalities and Fundamental Nonconvexities: A Reconciliation of Approaches to General Equilibrium Externality Modeling and Implications for Decentralization

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Abstract

By distinguishing between producible and nonproducible public commodities, we are able to propose a general equilibrium model with externalities that distinguishes between and encompasses both the Starrett [1972] and Boyd and Conley [1997] type external effects. We show that while nonconvexities remain fundamental whenever the Starrett type detrimental external effects are present, these are not caused by the type discussed in Boyd and Conley. On the basis of their disposability properties (costly or costless), the paper also distinguishes between by-product (such as pollution) and joint-product (such as national defense) producible public commodities and formally demonstrates market failure in the case of both beneficial and detrimental by-product producible public commodities. In contrast to price-based mechanisms, such as the Pigovian tax mechanism, it is shown that the notion of a "public competitive equilibrium" found in Foley [1967, 1970] provides a partially decentralized mechanism for economies with externalities that combines price and quantity signals with a unanimity criteria and is able to restore the equivalence between equilibrium and Pareto efficiency even in the presence of nonconvexities.
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by

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1. Introduction

Arrow [1970] perceived the market failure associated with externalities as a problem of incomplete markets. He showed that the equivalence between a competitive equilibrium and a Pareto optimum can be restored if markets for external effects can be created.

However, employing Arrow's framework, where the commodity space is extended to include the rights to generate externalities as additional commodities, Starrett [1972] demonstrated that the presence of detrimental production externalities creates fundamental nonconvexities in the technology sets of firms.[1] As is well known, when the convexity assumption fails, the existence of a competitive equilibrium becomes questionable.

A question then arises about the possibility of existence of some other alternative decentralized or *partially* decentralized mechanism that will, in the presence of externalities, ensure the equivalence between the underlying equilibrium concept and Pareto optimality.[2] In general, Hurwicz [1999] shows the impossibility of the existence of finite-dimensional decentralized mechanisms that guarantee Pareto optimality in the presence of externalities, for all economic environments (including nonconvex ones).

A popular candidate among the partially decentralized mechanisms with externalities is the one associated with Pigovian taxes, attributable to Pigou [1932] and Baumol [1972]. As has been well documented, an equilibrium with Pigovian taxes is compatible with nonconvex technology sets of the firms facing detrimental externalities, so long as the technologies of these firms are convex in the appropriate subspaces.[3] With full information about the economic environment, the planner can identify the level of the Pigovian taxes that equate the social marginal benefits to the social marginal costs of the externalities. However, the problem with the Pigovian tax mechanism is that, while any Pareto optimum can be decentralized as a Pigovian tax equilibrium, the reverse is not true. Baumol and Bradford [1972] showed that, if the detrimental effects of externalities on victim firms are sufficiently large, the *aggregate* technology set of the economy could well be nonconvex. In such a nonconvex economy, although the first order conditions of Pareto optimality would hold at a Pigovian tax equilibrium, the second order conditions for even a local Pareto optimum may fail. At

[1] He considers an example where increases in the level of an externality reduces the maximum output a firm can produce, given the levels of all inputs. But, the maximum output of the firm, for any given level of inputs, is assumed never to fall below zero, even in the face of an unlimited amount of the externality (the firm always has the option of shutting down production). This implies that the frontier of the technology is either asymptotic to the axis reserved for the externality or coincides with it after a critical level of the externality, where the maximum output has fallen to zero, has been reached.

[2] A mechanism is *decentralized* if the response of any agent to messages or signals received depends only on that agent's characteristics. By a *partially decentralized* mechanism we mean one where government is also an agent, but whose responses depend on the characteristics of a subset of all agents.

[3] See, *e.g.,* Starrett [1972], Baumol and Bradford [1972], and Hurwicz [1999].

these stationary points, there may exist adjustments in the underlying allocations that are Pareto improving, that is, if put to a vote, such adjustments would be unanimously accepted by all consumers.Thus, an arbitrary Pigovian tax equilibrium may not be efficient (*i.e.,* the first-welfare theorem analogue for a Pigovian tax equilibrium fails), *unless*, we restrict the class of economies to those where the externalities are weak enough to ensure convexity of the social transformation set, as is done in Hurwicz [1999].

More recently, however, Boyd and Conley [1997] (henceforth BC), and Conley and Smith [2002] have challenged the fundamentality of nonconvexities for *real* economies with externalities. They argue that nonconvexities are fundamental to the Arrow/Starrett (henceforth, AS) framework because it does not seem to offer a method of placing reasonable bounds on the extent to which the victim firms can observe the externality (sell externality rights to the generators). They argue that, in real economies, there are natural limits to the extent to which externalities can be generated. For example, the capacity of land, water, and air to absorb wastes and pollution is really not unlimited. They show that, in a model where the externality absorption capacity of the economy is a bounded resource, which has different qualitative but positive values for different agents, nonconvexities with externalities are no longer fundamental and a Coasian equilibrium can be defined, which will satisfy both the welfare theorems.

This paper aims to make two contributions. Firstly, we propose a general equilibrium model with externalities, which distinguishes between and encompasses both the AS and BC type external effects. This is done by distinguishing between producible and nonproducible public commodities.[4] We show that the objects of concern in BC are nonproducible public commodities, which are of positive value to agents. As BC argue, they do not lead to fundamental technological nonconvexities and markets will not fail if the aggregate endowment of these resources is bounded. AS's concern, on the other hand, is with the producible public commodities, whose availability can be enhanced above the initial endowment availability by production. We show that Starrett type technological nonconvexities will remain fundamental when detrimental producible public commodities are present. However, it will also be seen that nonconvexities in the technologies of the *victim* firms will not, on their own, imply market failure. Recalling that these are *producible* goods, market failure results when these nonconvexities are combined with some peculiarities in the production technologies of the *generating/externality producing* firms. From the generating firm's point of view, we show that the disposal of producible public commodities may be *free* (as *e.g.,* in the case of national defense) or *costly* (as *e.g.,* in the case of pollution). The paper shows that, abstracting from incentive problems, markets need not fail in the case of producible public commodities that are freely disposable, but will fail in the case of producible public commodities whose disposal is costly. This is true for *both* beneficial and detrimental externalities.

A second objective of this paper is to find a partially decentralized mechanism that restores the equivalence between equilibrium and efficiency (*i.e.,* satisfies both welfare theorems) even in the presence of nonconvexities caused by the AS type external effects. We find that, in contrast to the Pigovian-tax equilibrium, the "public competitive equilibrium" concept developed by Foley [1967, 1970] using Wicksell's [1896] initial ideas, when adapted to the context of externalities, will always be efficient even in the presence of nonconvexities.

The remaining paper is arranged as follows. In Section 2, we distinguish between producible and nonproducible public commodities and set up a general equilibrium model of

[4] In this paper, commodities that are nonrival in consumption (jointly consumed) will be called public commodities. These could be public goods such as national defense or public bads such as pollution.

externalities, which includes both AS and BC external effects. Section 3 discusses the non-convexities and market failures associated with certain producible public commodities. In Section 4, we define a collective consumption equilibrium and prove its welfare properties. We conclude in Section 5.

2. A Model of Producible and Nonproducible Public Goods.

An objective of this paper is to clarify the relation between BC and AS external effects. In an attempt to do so, we offer a taxonomy of public commodities that classifies these goods into nonproducible and producible public commodities. Nonproducible public commodities are those whose supply is fixed by the availability of their resources, while producible public commodities are those whose supply can be augmented by production beyond their respective resource availability.

The BC model attributes external effects to missing markets of some natural resources, such as air, resulting in an inefficient allocation of these resources. Such natural resources have competing uses. They can be valued in their pristine state, in which capacity they are often jointly consumed as public commodities, or they can be used as inputs to dump certain outputs of firms such as smoke. In the latter capacity, they possess the rivalness property (*e.g.,* the total amount of the air resource used up for dumping is the sum of individual amounts used for dumping by firms.) Moreover, it is reasonable to assume (as BC do) that such resources are bounded, so that the more they are used in their pristine states, the less is available for dumping. Thus, (the rights over) these goods are merely *exchanged* between agents, *i.e.*, their availability cannot be enhanced by production. Dumping agents who demand rights in excess of their endowments are net polluters, while dumping agents who demand less than their endowment of rights are abaters in BC. Consumers, in their model (see definition of a feasible allocation in BC, p. 399), *jointly* value these resources in their pristine states.[5] In a feasible state of the economy (*e.g.*, in an economic equilibrium) there is *no* positive net production of these goods. It is these goods that our taxonomy calls nonproducible public commodities.

Here is a clarifying example:[6] Water resource is bounded in a static world. Suppose this resource can be allocated between two alternative uses–it can be used as a recreational lake created by damming the water or it can be used as a dump for industrial wastes. The recreational lake is a public commodity, and raising the level of the dam implies a greater diversion of water for recreational purposes from dumping purposes. Thus, the water resource is a nonproducible public commodity in this example.

Public commodities whose supply can be augmented by production, are what we call producible public commodities. These include the standard public commodities such as national defense, bridges, highways, etc., which are freely disposable in production. These also include public commodities such as various noxious gases emited by firms, whose disposal in production is costly. Note that the production of these commodities may require the services of nonproducible public goods, *e.g.,* the disposal service of air is used to dump various noxious gases such as carbon dioxide, sulphur dioxide, nitrogen dioxide etc. Each of these gases has its own *distinct* disutility to all consumers (*e.g.,* each could have its own distinct impact on

[5] This is much in the spirit of time endowment, which is usually assumed to be fixed in general equilibrium models and allocated between firms and consumers. Firms value time endowment as a labor input into production, while consumers value time endowment in the form of leisure. There is zero net production of time.

[6] The author is grateful to an anonymous referee for this example.

health of consumers) and is a public bad.[7]

Moreover, a firm can produce infinite amounts of these public goods/bads provided the input requirements of both ordinary commodities and the nonproducible public commodities imposed by its technology are met. Of course, in a feasible state (or in an economic equilibrium), production of these public commodities will be bounded by the availability of the natural resources and other inputs of production. However, in a feasible state, there *can* be a positive net production of these goods over and above the initial endowments of these goods. The AS external effects are a result of absence of markets for producible public commodities. In particular, in their models, consumption of private goods by any agent produces public commodity externalities for all the other agents. It is as if the initial endowments of these public commodities was zero, and production enhances the supply above the endowment levels.

The index set of ordinary commodities (pure private goods possessing the rivalness property) is denoted by O. Vectors of ordinary commodities will be indexed by o. The index set for nonproducible public commodities will be N and vectors of these commodities will be indexed by n. The index set for producible public commodities will be G and vectors of these commodities will be indexed by g. We will assume that the cardinalities of sets O, N and G are also O, N, and G, respectively.

The endowment vector of ordinary commodities is denoted by ω_o and of nonproducible public commodities is denoted by $\omega_n + \sigma_n$. The initial stock of producible public commodities is ω_g. We assume that $\omega_o \in \mathbf{R}^O_+$, $\omega_n + \sigma_n \in \mathbf{R}^N_+$, and $\omega_g \in \mathbf{R}^G_+$.

There are four types of agents in the economy: (i) consumers, who are indexed by h that belongs to an index set H, (ii) firms for which the goods in N and G are public commodities. These are indexed by i that belongs to an index set I, (iii) firms for which goods in N are standard inputs (having rivalness property: the total use by all firms of these goods is the sum of individual uses), and which produce the goods in G. These are indexed by j that belongs to an index set J, and (iv) firms that value the goods in N in their pristine states, use these goods as inputs for disposal, and can both produce and consume goods in G. These are indexed by f belonging to an index set F.[8]

For all $i \in I$, the technology of firm i is denoted by $Y^i \subseteq \mathbf{R}^O \times \mathbf{R}^N_+ \times \mathbf{R}^G_+$, and its production vector is denoted by $y^i = \langle y^i_o, y^i_n, y^i_g \rangle \in \mathbf{R}^O \times \mathbf{R}^N_+ \times \mathbf{R}^G_+$. The net production of ordinary commodities is $y^i_o \in \mathbf{R}^O$. The consumption (use) of nonproducible public commodities by i in their pristine states is denoted by $y^i_n \in \mathbf{R}^N_+$, and its consumption of producible public commodities is denoted by $y^i_g \in \mathbf{R}^G_+$.

For all $j \in J$, the technology of firm j is denoted by $Z^j \subseteq \mathbf{R}^O \times \mathbf{R}^N_- \times \mathbf{R}^G$ and its net output vector is denoted by $z^j = \langle z^j_o, z^j_n, z^j_g \rangle \in \mathbf{R}^O \times \mathbf{R}^N_- \times \mathbf{R}^G$. The net production of ordinary commodities is $z^j_o \in \mathbf{R}^O$. The use by firm j of the nonproducible goods as inputs for dumping is $z^j_n \in \mathbf{R}^N_-$. The net addition to the stock of producible public commodities by firm

[7] The use of natural resources for dumping various types of outputs is similar to the input of (nonproducible) labor time being allocated to produce several (producible) goods such as apples and oranges by the firms. Each of such producible goods has its own distinct utility to consumers. There can be positive net production of these goods in a feasible state.

[8] In what follows the consumption of nonproducible public goods in their pristine state and the consumption of producible public goods by consumers in H and firms in I and F will take nonnegative values. This is because these goods are jointly consumed by firms in I and F along with consumers in H and, for the latter agents, the gross consumption sets are, by convention, subsets of the nonnegative orthant.

j is denoted by $z_g^j \in \mathbf{R}^G$.

We follow Milleron's [1972] notation to describe production of firms in F. For all $f \in F$, the technology of firm f is denoted by $W^f \subseteq \mathbf{R}^O \times \mathbf{R}^N_+ \times \mathbf{R}^N_- \times \mathbf{R}^G_+ \times \mathbf{R}^G$, and its production vector is denoted by $w^f = \langle w_o^f, w_{n+}^f, w_{n-}^f, w_{g-}^f, w_{g+}^f \rangle \in \mathbf{R}^O \times \mathbf{R}^N_+ \times \mathbf{R}^N_- \times \mathbf{R}^G_+ \times \mathbf{R}^G$. The net production of ordinary commodities is $w_o^f \in \mathbf{R}^O$. The consumption of goods in N in their pristine states by f is denoted by $w_{n+}^f \in \mathbf{R}^N_+$, the use of goods in N as inputs for dumping by f is denoted by $w_{n-}^f \in \mathbf{R}^N_-$, the use of public commodities in G as inputs by f is denoted by $w_{g-}^f \in \mathbf{R}^G_+$, and the net production of public commodities in G by f is denoted by $w_{g+}^f \in \mathbf{R}^G$, *e.g.*, a firm producing highways may be also consuming the services of *all* highways in the economy as inputs into its production.[9]

The consumption set of consumer $h \in H$ is denoted by $X^h \subseteq \mathbf{R}^{O+N+G}_+$. A consumption bundle is denoted by $x^h = \langle x_o^h, x_n^h, x_g^h \rangle \in \mathbf{R}^{O+N+G}_+$, where x_o^h is the gross consumption of ordinary commodities by h and x_n^h and x_g^h are the consumption levels of nonproducible and producible public commodities, respectively, by h. The preferences are represented by real valued continuous utility function u^h, for all $h \in H$.

An economy with above specifications will be represented by $E = \langle (X^h, u^h)_h, (Y^i)_i, (Z^j)_j, (W^f)_f, \omega_o, \omega_n + \sigma_n, \omega_g \rangle$.

Definition: A feasible allocation of the economy $E = \langle (X^h, u^h)_h, (Y^i)_i, (Z^j)_j, (W^f)_f, \omega_o, \omega_n + \sigma_n, \omega_g \rangle$ is a tuple $\langle (x^h)_h, (y^i)_i, (z^j)_j \rangle$ such that $x^h \in X^h$ for all $h \in H$, $y^i \in Y^i$ for all $i \in I$, $z^j \in Z^j$ for all $j \in J$, $w^f \in W^f$ for all $f \in F$, and

$$\begin{aligned} \sum_{h \in H} x_o^h = \sum_{i \in I} y_o^i + \sum_{j \in J} z_o^j + \sum_{f \in F} w_o^f + \omega_o,\ x_n^h = \sum_{j \in J} z_n^j + \sum_{f \in F} w_{n-}^f + \omega_n + \sigma_n \ \forall h \in H, \\ y_n^i = \sum_{j \in J} z_n^j + \sum_{f \in F} w_{n-}^f + \omega_n + \sigma_n \ \forall i \in I,\ w_{n+}^f = \sum_{j \in J} z_n^j + \sum_{f \in F} w_{n-}^f + \omega_n + \sigma_n \ \forall f \in F, \\ x_g^h = \sum_{j \in J} z_g^j + \sum_{f \in F} w_{g+}^f + \omega_g \ \forall h \in H,\ y_g^i = \sum_{j \in J} z_g^j + \sum_{f \in F} w_{g+}^f + \omega_g \ \forall i \in I, \text{ and} \\ w_{g-}^f = \sum_{j \in J} z_g^j + \sum_{f \in F} w_{g+}^f + \omega_g \ \forall f \in F. \end{aligned} \tag{2.1}$$

Firms in F were introduced to show the generality of the model in being able to include bilateral externalities.[10] An objective of this paper is to propose a partially decentralized mechanism with externalities, which will result in a Pareto efficient allocation. The mechanism proposed in this paper distinguishes between the producers and the consumers of external effects. It involves a creation of a collective action of victims who collectively arrange the purchase or sale of externality rights to the generators. In the case of unilateral externalities, there is a separation between the victims and the generators of externalities. In the case of bilateral externalities, there is no such separation as an agent can be both a victim and a generator of external effects. However, even in this case a version of the proposed mechanism can be defined, where agents in their capacity as victims form a collective action and transact

[9] The author is grateful to an anonymous referee for this example.

[10] More generally, it is possible, in an exactly analogous way, to allow for consumption externalities as well. In the interest of space and notation, we refrain from doing so here. The model can also be easily extended to encompass externalities created by absence of markets for goods having the rivalness feature.

with themselves in their capacity as generators of external effects. The proposed mechanism will be intuitively less appealing in the case of bilateral externalities and also notationally more complex. For these reasons, we will exclude firms in F from our analysis. Restricting focus to studying unilateral externalities is not uncommon in the literature, see for example, Coase [1960], Pigou [1920], Hurwicz [1999] and BC.[11]

We now provide a private ownership structure for the endowments in this economy that can sustain a market based institutional structure with public commodities. We assume that the endowment of all private goods is owned by the consumers in H. Let ω_o be distributed as $(\omega_o^h)_h$. The endowment of nonproducible public commodities is divided between those for whom they are public commodities and those for whom they are standard (rival) inputs. Because they are public commodities for agents in H and I, we assume that each agent in H and I owns the same amount ω_n of such resources. The remaining amount σ_n is distributed between agents in J as $(\sigma_n^j)_j$. Lastly, we assume that each agent in H and I owns the same amount ω_g of the initial stock of producible public commodities.[12]

We define the restrictions of the technology sets to appropriate subspaces by employing the following notation: For all $i \in I$ and $y_g^i \in \mathbf{R}_+^G$, define $Y^i(y_g^i) := \{\langle y_o^i, y_n^i\rangle \in \mathbf{R}^O \times \mathbf{R}_+^N \,|\, \langle y_o^i, y_n^i, y_g^i\rangle \in Y^i\}$. Likewise, we can define for all $j \in J$, $Z^j(z_o^j, z_g^j)$, and so on. For any vector $a \in \mathbf{R}^S$, the s^{th} element of a will be denoted by $a(s)$. Typical elements of vectors of ordinary commodities, nonproducible public goods, and producible public goods will be denoted by α, β, and γ, respectively. Thus, for example, $x_o^h(\alpha)$ is the amount of the α^{th} ordinary commodity consumed by $h \in H$. Sometimes this notation will also be used to denote sets which are defined for particular commodities, *e.g.,* $\mathcal{W}_o(\alpha)$ is a set pertaining to $\alpha \in O$.

3. Nonconvexities and Market Failures.

3.1. Nonconvexities.

In BC, all users of the nonproducible public commodities have positive value for these goods. In particular, consumers in H and firms in I have positive value for these resources in their pristine states. This rules out nonconvexities in the preferences and technologies of these agents.[13]

In the context of producible public commodities, Starrett and Otani and Sicilian [1977] show that technology sets of firms facing detrimental externalities are nonconvex. We demonstrate that the restricted profit functions of firms in I, for whom some goods in G are detrimental, are nonconcave.[14] Restricted profit functions will be later employed to define a col-

[11] Although Conley and Smith [2002] generalize BC to allow for consumption externalities and externalities flowing from firms to firms. However, the Coasian equilibrium they define is viz-a-viz nonproducible public commodities in our model. As demonstrated by Starrett and as will be examined later, a Coasian equilibrium may not exist when producible public commodities are also present.

[12] We assume collective ownership of ω_n and ω_g in view of the equilibrium concept we will study later, which involves a collective action of victims. However, like in BC, it is possible to assume that ω_n and ω_g are distributed among consumers in H and firms in I.

[13] However, Theorem 1, below, will continue to apply if a firm in I has a negative value for some nonproducible public good, *e.g.,* proximity to a forest cover, a nonproducible public commodity in the short run, may be valued negatively by a laundry requiring sun-light.

[14] The case of nonconvexities in preferences due to consumption externalities can also be handled along the axiomatic line developed in this section. See, for example, an earlier version (Murty [2006]) of this paper.

lective consumption equilibrium. Further, the nonconcavity of these functions will be useful in demonstrating, formally, market failure for a category of producible public commodities.

Definition: Good $\gamma \in G$ is a *detrimental (beneficial)* producible public commodity for firm $i \in I$ if there exist $\bar{y}_g^i, y_g^i \in \mathbf{R}_+^G$ with $\bar{y}_g^i(\gamma) < y_g^i(\gamma)$ $(\bar{y}_g^i(\gamma) > y_g^i(\gamma))$ and $\bar{y}_g^i(\gamma') = y_g^i(\gamma')\ \forall \gamma' \in G$ and $\gamma' \neq \gamma$, such that $Y^i(y_g^i) \subseteq Y^i(\bar{y}_g^i)$ and *Boundary* $(Y^i(y_g^i)) \cap$ *Boundary* $(Y^i(\bar{y}_g^i)) = \emptyset$.

Intuitively, $\gamma \in G$ is a detrimental (beneficial) public commodity if there exist increases in its level that contract (expand) the set of production possibilities of all other commodities. For firms in I, we introduce two assumptions.

Assumption (FED) Finite Externality Damage: $\bigcap_{y_g^i \in \mathbf{R}_+^G} Y^i(y_g^i) \neq \emptyset$.

Assumption (CERT) Closed Externality Restricted Technology: $Y^i(y_g^i)$ is closed $\forall\ y_g^i \in \mathbf{R}_+^G$.

On the one hand, Assumption (FED) provides a general unboundedness condition on the technologies of firms in I in the direction of goods in G (both beneficial and detrimental) and, on the other, is a generalization of Starrett's view that the damage done by detrimental producible public commodities is finite: precisely, in his model, shutting down production of ordinary commodities is an option available to the victim firm at every level of the production externality observed. Thus, this is a compelling assumption on the technologies of firms in I. Assumption (CERT) implies that the technology conditional on any level of goods in G is closed.

For all $y_g^i \in \mathbf{R}_+^M$, let $B_y^i(y_g^i)$ denote the barrier cone of $Y^i(y_g^i)$.[15] Define the restricted profit function of $i \in I$ as a function of $y_g^i \in \mathbf{R}_+^G$ and $\langle p_o, p_{y_n}^i \rangle \in B_y^i(y_g^i)$ as

$$\pi = \hat{\Pi}_y^i(y_g^i, p_o, p_{y_n}^i) := \max_{y_o^i, y_n^i} \{p_o \cdot y_o^i + p_{y_n}^i \cdot [y_n^i - \omega_n] \big| \ \langle y_o^i, y_n^i \rangle \in Y^i(y_g^i)\}. \tag{3.1}$$

The argmax of (3.1) is denoted by the function

$$\langle y_o^i, y_n^i \rangle = \langle y_o^i(y_g^i, p_o, p_{y_n}^i),\ y_n^i(y_g^i, p_o, p_{y_n}^i) \rangle. \tag{3.2}$$

Assumptions (FED) and (CERT) combined with some further assumptions (strong continuity and semi-boundedness of the correspondence $Y^i(y_g^i))$ translate into the following properties for $\hat{\Pi}_y^i$.[16]

Remark 1. If $\gamma \in G$ is a detrimental (beneficial) producible public commodity, then there exist $\bar{y}_g^i, \tilde{y}_g^i \in \mathbf{R}_+^G$ with $\bar{y}_g^i(\gamma) < \tilde{y}_g^i(\gamma)$ and $\bar{y}_g^i(\gamma') = \tilde{y}_g^i(\gamma'), \forall\ \gamma' \in G$ and $\gamma' \neq \gamma$ such that $\hat{\Pi}_y^i(\tilde{y}_g^i, p_o, p_{y_n}^i) < \hat{\Pi}_y^i(\bar{y}_g^i, p_o, p_{y_n}^i)$ $(\hat{\Pi}_y^i(\tilde{y}_g^i, p_o, p_{y_n}^i) > \hat{\Pi}_y^i(\bar{y}_g^i, p_o, p_{y_n}^i))$. Further, $\hat{\Pi}_y^i$ is continuous in $y_g^i \in \mathbf{R}_+^G$.[17]

Maintaining the assumptions of strong continuity and semi-boundedness of the correspondence $Y^i(y_g^i)$, the following theorem shows that the hypograph of the restricted profit function in the space of y_g^i and π is nonconvex.[18]

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15 As defined by McFadden [1978], the barrier cone $B_y^i(y_g^i)$ of a set $Y^i(y_g^i)$ is the set of all prices $\langle p_o, p_{y_n}^i \rangle$ such that the set $\{p_o \cdot o^i + p_{y_n}^i \cdot y_n^i \big| \langle y_o^i, y_n^i \rangle \in Y^i(y_g^i)\}$ is bounded above.

16 See McFadden [1978] pp. 62-65 for the definitions and implications of strong continuity and semi-boundedness of $Y^i(y_g^i)$. These assumptions are satisfied, *e.g.,* when technology is irreversible and when the correspondence $Y^i(y_g^i)$ is convex and continuous.

17 See McFadden and Fuss [1978].

18 Let $a \in \mathbf{R}^n$. We denote the ϵ neighborhood of $a \in \mathbf{R}^n$ by $\mathcal{N}_\epsilon(a) \subseteq \mathbf{R}^n$.

Theorem 1: *For all $i \in I$,*

(1) under Assumption (FED), given any $\langle p_o, p^i_{y_n}\rangle \in \cap_{y^i_g \in \mathbf{R}^G_+}$ int $B^i_y(y^i_g)$, there exists $c \in \mathbf{R}$ such that $\hat{\Pi}^i_y(y^i_g, p_o, p^i_{y_n}) \geq c$ for all $y^i_g \in \mathbf{R}^G_+$, and

(2) if $\gamma \in G$ is a detrimental producible public commodity, $Y^i(y^i_g)$ satisfies strong continuity and semi-boundedness, and Assumptions (FED) and (CERT) hold, then for all $\langle p_o, p^i_{y_n}\rangle \in \cap_{y^i_g \in \mathbf{R}^G_+}$ int $B^i_y(y^i_g)$, the set $\{\langle y^i_g, \pi\rangle \in \mathbf{R}^{G+1}_+ | \pi \leq \hat{\Pi}^i_y(y^i_g, p_o, p^i_{y_n})\}$ is nonconvex.

Proof:
(1) Under Assumption (FED), $\exists\langle \bar{y}^i_o, \bar{y}^i_n\rangle \in \bigcap_{y^i_g \in \mathbf{R}^G_+} Y^i(y^i_g)$. Choose any $\langle p_o, p^i_{y_n}\rangle$ $\in \cap_{y^i_g \in \mathbf{R}^G_+}$ *int* $B^i_y(y^i_g)$. Define $c = p_o \cdot \bar{y}^i_o + p^i_{y_n} \cdot \bar{y}^i_n$. c has the required property from the definition (3.1) of $\hat{\Pi}^i_y$.
(2) Suppose not. Then for all $\langle p_o, p^i_{y_n}\rangle \in \cap_{y^i_g \in \mathbf{R}^G_+}$ *int* $B^i_y(y^i_g)$, we have $A := \{\langle y^i_g, \pi\rangle \in \mathbf{R}^{G+1}_+ | \pi \leq \hat{\Pi}^i_y(y^i_g, p_o, p^i_{y_n})\}$ is convex. Since γ is a detrimental producible public commodity, Remark 1 implies that $\exists \bar{y}^i_g, \tilde{y}^i_g \in \mathbf{R}^G_+$ with $\bar{y}^i_g(\gamma) < \tilde{y}^i_g(\gamma)$ and $\bar{y}^i_g(\gamma') = \tilde{y}^i_g(\gamma'), \forall\ \gamma' \in G$ and $\gamma' \neq \gamma$ such that $\tilde{t} := \hat{\Pi}^i_y(\tilde{y}^i_g, p_o, p^i_{y_n}) < \hat{\Pi}^i_y(\bar{y}^i_g, p_o, p^i_{y_n}) =: \bar{t}$. From Remark 1, $\hat{\Pi}^i_y$ is continuous in y^i_g. Hence, $\exists \epsilon > 0$ and $\delta > 0$ such that $\hat{\Pi}^i_y(y^i_g, p_o, p^i_{y_n}) \in \mathcal{N}_\epsilon(\bar{t})$ whenever $y^i_g \in \mathcal{N}_\delta(\bar{y}^i_g)$ and $\tilde{t} < t\ \forall t \in \mathcal{N}_\epsilon(\bar{t})$. From (1) of this theorem $\exists c \in \mathbf{R}$ such that $\langle y^i_g, c\rangle \in A\ \forall y^i_g \in \mathbf{R}^G_+$. Choose $\lambda \in [0,1]$ such that $\overset{*}{t} := \lambda c + (1-\lambda)\bar{t} \in \mathcal{N}_\epsilon(\bar{t})$. We can freely choose $\overset{*}{y}{}^i_g \in \mathbf{R}^G_+$ big enough such that $\tilde{y}^i_g = \lambda \overset{*}{y}{}^i_g + (1-\lambda)\bar{y}^i_g$. Hence, by maintained convexity of set A, we have that $\langle \tilde{y}^i_g, \overset{*}{t}\rangle \in A$. But this means $\overset{*}{t} \leq \tilde{t}$. This contradicts $\overset{*}{t} \in \mathcal{N}_\epsilon(\bar{t})$.■

3.2. Market Failure.

In a model with only nonproducible public commodities, which give positive value to all users, BC demonstrated that markets need not fail. They define a Coasian equilibrium and show that it is Pareto optimal and a Pareto optimum can be decentralized as a Coasian equilibrium. Conley and Smith [2002] prove the existence of a Coasian equilibrium for a convex economy.

To study the possibility of market failure in the case of producible public commodities, we distinguish between two types of producible public commodities: by-product and joint-product producible public commodities. This classification is based on the nature of the technologies of the externality generating firms (firms in J). Standard disposability conditions are violated in the case of by-product producible public commodities, such as pollution. Rather, their disposal is costly.[19] This would imply that profit maximization for the generating firms is well defined only if generators *pay* a positive price for producing these goods. However, from the pollutees' point of view, receiving positive prices for the externalities implies an infinite demand for such goods (both beneficial and detrimental) and hence, the market failure. This is demonstrated in Theorem 2 below.

In contrast joint-product producible public commodities, such as national defense, bridges, etc. are freely disposable. Profit maximization is well defined only if the the generators *receive* a positive price to produce such goods. This means that the users of such goods must pay a positive price to consume such goods, which they are willing to pay if these goods are beneficial for them. If such goods have negative value for the users (implying nonconvexities

[19] See definitions below and Remark 2.

in the technologies of firms in I) they will have zero demands for such goods at positive prices paid. So, abstracting from issues such as free-riding and thin markets, markets need not fail for these goods.

Suppose the index set of producible external effects is partitioned into the sets G_1 and G_2. Likewise, we partition any vector of external effects into $z_{g_1} \in \mathbf{R}^{G_1}$ and $z_{g_2} \in \mathbf{R}^{G_2}$.

Assumption (CD1) (Costly Disposal 1): For all $\langle z_o^j, z_n^j, z_{g_2}^j \rangle \in \mathbf{R}^O \times \mathbf{R}_-^N \times \mathbf{R}^{G_2}$, the sets $Z^j(z_o^j, z_n^j, z_{g_2}^j)$ are bounded below.

Assumption (CD2) (Costly Disposal 2): For all $\langle z_o^j, z_n^j, z_{g_1}^j, z_{g_2}^j \rangle \in Z^j$ if $\overset{*}{z}{}_{g_1}^j \geq z_{g_1}^j$, then $\langle z_o^j, z_n^j, \overset{*}{z}{}_{g_1}^j, z_{g_2}^j \rangle \in Z^j$.

Assumptions (CD1) and (CD2) are postulated to capture externalities like pollution. They are two alternative assumptions that reflect the fact that the disposal of these goods is costly. (CD1) implies, for example, that there is a minimum level of smoke that can be produced from any given level of coal and (CD2) says that if a certain quantity of coal produces some level of smoke, then it can also produce any level of smoke higher than it. It can be argued that (CD1) captures the spirit of costly disposability of goods such as pollution better than (CD2). But note that (CD2) is a polar opposite of the standard free-disposability assumption. Indeed, it can be shown that if an equilibrium based on profit maximization in an economy where (CD2) is satisfied exists, then the firms in J will be producing at the lower bounds defined by (CD1). We use (CD2) to demonstrate market failure in the case of by-product externalities.[20]

Definition. For all $j \in J$, the index set $G_1 \subseteq G$ *comprises of by-products of* Z^j if Assumptions (CD1) or (CD2) are true.

On the other hand, joint-product producible public commodities are outputs of firms in J whose disposal is free. These include the standard public commodities. Assumption (FD) is such a free disposability condition.

Assumption (FD) (Free Disposal): For all $\langle z_o^j, z_n^j, z_{g_1}^j, z_{g_2}^j \rangle \in Z^j$, if $\overset{*}{z}{}_{g_2}^j \leq z_{g_2}^j$, then $\langle z_o^j, z_n^j, z_{g_1}^j, \overset{*}{z}{}_{g_2}^j \rangle \in Z^j$.

Definition. For all $j \in J$, the index set $G_2 \subseteq G$ *comprises of joint-products of* Z^j if Assumption (FD) holds.

For all $j \in J$, suppose B_z^j denotes the barrier cone of Z^j. Define the unrestricted profit function of $j \in J$ for all $\langle p_o, p_{z_n}, p_{z_g} \rangle \in B_z^j$ as

$$\pi = \Pi_z^j(p_o, p_{z_n}, p_{z_g}) := \max_{z_o^j, z_n^j, z_g^j} \{p_o \cdot z_o^j + p_{z_n} \cdot (z_n^j + \sigma_n^j) + p_{z_g} \cdot z_g \big| \langle z_o^j, z_n^j, z_g^j \rangle \in Z^j\}. \tag{3.3}$$

Suppose the solution to (3.3) is given by

$$\langle z_o^j, z_n^j, z_g^j \rangle = \langle z_o^j(p_o, p_{z_n}, p_{z_g}),\ z_n^j(p_o, p_{z_n}, p_{z_g}),\ z_g^j(p_o, p_{z_n}, p_{z_g}) \rangle. \tag{3.4}$$

If γ is a by-product producible public commodity and $p_{z_g}(\gamma) > 0$, then (CD2) implies that profits of $j \in J$ are unbounded. Similarly, because of (FD), profits of $j \in J$ will be unbounded if γ is a joint-product producible public commodity and $p_{z_g}(\gamma) < 0$.

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[20] Another characteristic feature of producible public commodities like pollution is that their production is often correlated to the production of other commodities, *e.g.,* emission of green house gases is correlated to the use of fossil fuels, which are used in producing ordinary commodities. However, Assumptions (CD1) or (CD2) are not enough to capture this fact and need to be combined with another assumption that can capture such a correlation. See an earlier version (Murty [2006]) for the additional assumption of "Byproduction" that captures this correlation.

Remark 2. There will be no solution to (3.3) if price of joint-product producible public commodities is negative and price of by-product producible public commodities is positive when Assumption (CD2) is true.

For all $i \in I$, $p^i_{y_g} \in \mathbf{R}^G$, and $\langle p_o, p^i_{y_n}\rangle \in \cap_{y^i_g \in \mathbf{R}^G_+}$ *int* $B^i_y(y^i_g)$, define the unrestricted profit function

$$\pi = \Pi^i_y(p_o, p^i_{y_g}, p^i_{y_n}) := \max_{y^i_g \in \mathbf{R}^G_+} \{p^i_{y_g} \cdot (y^i_g - \omega_g) + \hat{\Pi}^i_y(y^i_g, p_o, p^i_{y_n})\}. \tag{3.5}$$

From Remark 2 above, firms in J must *pay* a postive price for producing $\gamma \in G_1$ (*i.e.,* $p_{z_g}(\gamma) < 0$). This implies that there exists a firm $i \in I$ that *receives* a positive price for consuming γ (*i.e.,* $p^i_{y_g}(\gamma) > 0$). The following theorem proves Starrett's conclusion that the profit of firm i is unbounded (*i.e.,* there is no solution to (3.5)) at positive price received for goods in G_1. The reason for this result is Assumption (FED), the unboundedness condition on technologies of firms in I when G_1 is not an empty set. Assumption (FED) is consistent with both detrimental and beneficial externalities. For the reasons explained in the beginning of this section, this demonstrates the market failure in the case of both beneficial and detrimental by-product public commodities.

Theorem 2: *Under Assumptions (FED) and (CERT), for all $i \in I$ and for all $\langle p_o, p^i_{y_n}\rangle \in \cap_{y^i_g \in \mathbf{R}^G_+}$ int $B^i_y(y^i_g)$, there is no solution to (3.5) if there exists $\gamma \in G_1$ such that $p^i_{y_g}(\gamma) > 0$.*

Proof: Conclusions of Remark 1 and Theorem 1 hold under Assumptions (FED) and (CERT). Note, problem (3.5) can equivalently be re-written as

$$\pi = \Pi^i_y(p_o, p^i_{y_g}, p^i_{y_n}) := -p^i_{y_g} \cdot \omega_g + \max_{\langle a^i, b^i\rangle \in \mathbf{R}^{G+1}_+} \{p^i_{y_g} \cdot a^i + b^i \big| b^i \leq \hat{\Pi}^i_y(a^i, p_o, p^i_{y_n})\}. \tag{3.6}$$

Suppose $\langle \overset{*}{a}^i, \overset{*}{b}^i\rangle$ solves (3.6). Then $\hat{\Pi}^i_y(\overset{*}{a}^i, p_o, p^i_{y_n}) = \overset{*}{b}^i$. By Conclusion (1) of Theorem 1 there exists $c \in \mathbf{R}$ such that $c \leq \hat{\Pi}^i_y(y^i_g, p_o, p^i_{y_n})$, $\forall y^i_g \in \mathbf{R}^G_+$. In particular, $c \leq \overset{*}{b}^i$. Partition G into $\mathcal{G}^i_1 := \{\gamma \in G_1 | p^i_{y_g}(\gamma) > 0\}$ and its complement relative to G, denoted by $\mathcal{G}^i_2$.[21] Accordingly, rearrange elements in vectors of external effects y^i_g and their prices $p^i_{y_g}$ to obtain $y^i_g = \langle y^i_{\mathcal{G}_1}, y^i_{\mathcal{G}_2}\rangle$ and $p^i_{y_g} = \langle p^i_{y\mathcal{G}_1}, p^i_{y\mathcal{G}_2}\rangle$.[22] Define $\bar{y}^i_g = \langle \bar{y}^i_{\mathcal{G}_1}, \overset{*}{a}^i_2\rangle$ such that $p^i_{y\mathcal{G}_1} \cdot \bar{y}^i_{\mathcal{G}_1} + p^i_{y\mathcal{G}_2} \cdot \overset{*}{a}^i_2 + c = \overset{*}{b}^i + p^i_{y_g} \cdot \overset{*}{a}^i$. Now freely choose $\tilde{y}^i_g = \langle \tilde{y}^i_{\mathcal{G}_1}, \overset{*}{a}^i_2\rangle$ such that $\tilde{y}^i_{\mathcal{G}_1} \gg \bar{y}^i_{\mathcal{G}_1}$ and $\tilde{y}^i_g \in \mathbf{R}^G_+$. Then by Conclusion (1) of Theorem 1, $c \leq \hat{\Pi}^i_y(\tilde{y}^i_g, p_o, p^i_{y_n})$ and $p^i_{y_g} \cdot \tilde{y}^i_g + c > p^i_{y_g} \cdot \bar{y}^i_g + c = \overset{*}{b}^i + p^i_{y_g} \cdot \overset{*}{a}^i$. A contradiction to $\langle \overset{*}{a}^i, \overset{*}{b}^i\rangle$ solves (3.6).■

4. Collective Consumption Equilibrium.

The previous section demonstrated the failure of competitive markets for goods in G_1: precisely, a market equilibrium cannot exist even if property rights were well established. This is immaterial of whether these goods generate detrimental or beneficial externalities. If some goods in G_1 create detrimental externalities, then we have the problem of technological

[21] Note that $G_2 \subseteq \mathcal{G}^i_2$.

[22] Clarifying further this notation, the elements of $y^i_{\mathcal{G}_1}$ would be the level of public commodities in G_1 with positive prices.

nonconvexities and we are confronted with Hurwicz's negative result for the existence of non-wasteful decentralized mechanisms. Turning to partially decentralized mechanism, as we discussed in the introduction, any such mechanism based purely on transmission of information regarding shadow prices, such as the Pigovian tax mechanism, will not ensure efficiency of the equilibrium in nonconvex economies.

We find that the concept of a "public competitive equilibrium" developed by Foley [1967, 1970] based on the ideas of Wicksell [1896], can be adapted to the context of externalities created by producible public commodities. Precisely because his mechanism allows a degree of decentralized choice by combining both price and quantity signals and applying a unanimity criteria, it permits a concept of an equilibrium for general (including nonconvex) economies that will always be efficient.

The equilibrium that will be defined in this section involves a separation of the finance from the production of public commodities: an agency (perhaps the government) is formed via the collective action of individual consumers to manage their collective needs of the public commodity. This agency takes the market prices of the public commodities p_{z_g} as given and collects contributions from consumers to finance the purchase of a vector of these goods c_g from the producers.[23] Denote a price system by the vector $p = \langle p_o, (p^h_{x_n})_{h\in H}, (p^i_{y_n})_{i\in I}, p_{z_n}, p_{z_g}\rangle$, where the consumers and firms in I participate in the markets for the commodities in O and N and the vector of prices faced by consumer h is denoted by $\langle p_o, p^h_{x_n}\rangle$.[24]

Definition. A *government proposal relative to the price system* p is a profile of individual lump-sum taxes for consumers $\mathcal{T} := (\mathcal{T}^h)_h \in \mathbf{R}^H$ and levels of externalities $c_g \in \mathbf{R}^G_+$ such that $\sum_h \mathcal{T}^h = p_{z_g}(c_g - \omega_g)$. It is denoted by $\langle \mathcal{T}, c_g\rangle$.

We derive a private ownership economy from the economy E. The structure of ownership of resources is as described in Section 2 and, for all $h \in H$, $i \in I$ and $j \in J$, $\theta^h_y(i)$ and $\theta^h_z(j)$ denote the shares of consumer h in the profits of firms i and j, respectively. The private ownership economy will be denoted by $E^P = \langle (X^h, u^h)_h, (Y^i)_i, (Z^j)_j, (\omega^h_o, \theta^h_y(i), \theta^h_z(j), \sigma^j_n)_{h,i,j}, \omega_n, \omega_g\rangle$.

For all $h \in H$, the upper level sets (strictly upper level sets) of u^h are denoted by $R^h(\overset{*}{u}) := \{\langle x^h_o, x^h_n, x^h_g\rangle \in X^h |\ u^h(x^h_o, x^h_n, x^h_g) \geq \overset{*}{u}\}$ $(P^h(\overset{*}{u}) := \{\langle x^h_o, x^h_n, x^h_g\rangle \in X^h |\ u^h(x^h_o, x^h_n, x^h_g) > \overset{*}{u}\})$. We also define restrictions of these sets to appropriate subspaces, *e.g.*, $R^h(\overset{*}{u}, x^h_g) := \{\langle x^h_o, x^h_n\rangle \in \mathbf{R}^{O+N}_+ |\ \langle x^h_o, x^h_n, x^h_g\rangle \in X^h$ and $u^h(x^h_o, x^h_n, x^h_g) \geq \overset{*}{u}\}$. Let $B^h_x(x^h_g)$ be the set of prices $\langle p_o, p^h_{x_n}\rangle \in \mathbf{R}^{O+N}$ such that the sets $\{p_o x^h_o + p^h_{x_n} x^h_n | \langle x^h_o, x^h_n\rangle \in R^h(u, x^h_g)\}$ are bounded below for all u.

The income of any $h \in H$ at a given price system p, a given distribution of profit shares $\langle (\theta^h_y(i))_{h,i}, (\theta^h_z(j))_{h,j}\rangle$, and a government proposal $\langle \mathcal{T}, c_g\rangle$ is

$$\begin{aligned} r^h(p, c_g, \mathcal{T}^h) = p_o\ \omega^h_o + p^h_{x_n}\omega_n + \sum_i \theta^h_y(i)\hat{\Pi}^i_y(p_o, p^i_{y_n}, c_g) \\ + \sum_j \theta^h_z(j)\Pi^j_z(p_o, p_{z_n}, p_{z_g}) - \mathcal{T}^h, \end{aligned} \tag{4.1}$$

where, for all $i \in I$ and $j \in J$, $\hat{\Pi}^i_y(p_o, p^i_{y_n}, c_g)$ and $\Pi^j_z(p_o, p_{z_n}, p_{z_g})$ are defined as in (3.1) and (3.3), respectively. For all $c_g \in \mathbf{R}^G_+$, $\mathcal{T} \in \mathbf{R}^{\mathbf{H}}$, and prices $\langle p_o, p^h_{x_n}\rangle \in B^h_x(c_g)$, the externality

[23] In the context of detrimental external effects, this amounts to the agency choosing the level of rights to pollute to sell to the generating firms and redistributing proceeds to consumers as transfers.

[24] p_{z_n} and p_{z_g} will be the common prices faced by the firms in J for commodities in N and G, respectively.

restricted indirect utility function of consumer $h \in H$ is defined by

$$V^h(p_o, p^h_{x_n}, c_g, r^h(p, c_g, \mathcal{T}^h)) := \max_{x^h_o, x^h_n} \{u^h(x^h_o, x^h_n, c_g) \big| \langle x^h_o, x^h_n, c_g\rangle \in X^h \text{ and } p_o x^h_o + p^h_{x_n} x^h_n \leq r^h(p, c_g, \mathcal{T}^h)\}. \tag{4.2}$$

Suppose the argmax of (4.2) is given by the functions

$$\langle x^h_o, x^h_n\rangle = \langle x^h_o(p_o, p^h_{x_n}, c_g, r^h(p, c_g, \mathcal{T}^h)), x^h_n(p_o, p^h_{x_n}, c_g, r^h(p, c_g, \mathcal{T}^h))\rangle. \tag{4.3}$$

If u^h and r^h are continuous, then V^h is continuous in its arguments.[25] Budget proposals relative to the current system of prices are offered to all consumers by the collective action. A budget proposal is accepted iff it is not unanimously rejected by all the consumers in favor of some other government budget proposal relative to the current system of prices.

Definition. A government budget proposal $\langle \mathcal{T}, c_g\rangle$ relative to price system p, is *unanimously rejected* in favor of another government budget proposal $\langle \overset{*}{\mathcal{T}}, \overset{*}{c}_g\rangle$ relative to price system p if $V^h(p, c_g, r^h(p, c_g, \mathcal{T}^h)) \leq V^h(p, \overset{*}{c}_g, r^h(p, \overset{*}{c}_g, \overset{*}{\mathcal{T}}{}^h))$, $\forall h \in H$ and $V^h(p, c_g, r^h(p, c_g, \mathcal{T}^h)) < V^h(p, \overset{*}{c}_g, r^h(p, \overset{*}{c}_g, \overset{*}{\mathcal{T}}{}^h))$ for some $h \in H$.

Definition. A *collective consumption equilibrium (CCE)* of $E^P = \langle (X^h, u^h)_h, (Y^i)_i, (Z^j)_j, (\omega^h_o, \theta^h_y(i), \theta^h_z(j), \sigma^j_n)_{h,i,j}, \omega_n, \omega_g\rangle$ is a configuration $\langle p, c_g, \mathcal{T}, (x^h_o)_h, (y^i_o)_i, (z^j_o)_j, (x^h_n)_h, (y^i_n)_i, (z^j_n)_j, (z^j_g)_j\rangle$ such that (i) $\langle (x^h_o, x^h_n)_h, (y^i_o, y^i_n)_i, (z^j_o, z^j_n, z^j_g)_j\rangle$ solve (4.3), (3.2), and (3.4), respectively, for $\langle p, c_g, \mathcal{T}\rangle$; (ii) $\langle c_g, \mathcal{T}\rangle$ is a government budget proposal relative to price system p that is not unanimously rejected in favor of any other government budget proposal relative to price system p; and (iii)[26]

$$\sum_h p^h_{x_n} - \sum_i p^i_{y_n} = p_{z_n},\ \sum_h x^h_o = \sum_i y^i_o + \sum_j z^j_o + \omega_o,\ c_g - \omega_g = \sum_j z^j_g, \text{ and } y^i_n - \omega_n = x^h_n - \omega_n = \sum_j (z^j_n + \sigma^j_n),\ \forall h \in H, i \in I. \tag{4.4}$$

Hammond and Villar [1998] have also explored the potential of applying Foley's ideas to study general public commodity economies where nonconvexities may emerge (for diverse reasons, *e.g.,* scale economies) and create problems for defining partially decentralized equilibrium concepts that will satisfy the first-welfare theorem. The government decides the level of the public commodities to be produced, taxes the consumers, and subsidizes the producers of the public commodity. They propose a "farsighted" public competitive equilibrium based on a cost-benefit test. However, the equilibrium concept they propose is different from the CCE, as a government proposal in their model is unanimously rejected if there exists another government proposal that results in a *price equilibrium*, conditional on the new proposed level of public commodities, that Pareto dominates the earlier proposal. In contrast, in a CCE, the collective action controls only the victims or the beneficiaries of the public commodity

25 See Diewert [1974].

26 The first equality indicates the implicit personalized markets for goods in N. It is as if each firm $j \in J$ participates in personalized markets with each $h \in H$ and $i \in I$ and pays, $p^h_{x_n}$ and $p^i_{y_n}$, respectively.

externalities, takes prices as given, and searches for a government proposal (and not a conditional equilibrium) that will not be Pareto dominated by another proposal at the existing prices. The proposal that is not overthrown at current prices, expresses the *collective demand* for the public commodity of the collective action at those prices. This demand is determined by aggregating over individual consumer preferences by using the unanimity criteria. The supply at the given prices is determined by the profit calculus of the generators of the public commodity externalities. As Malinvaud [1985] observes, "the economy will preserve some degree of decentralization with the consumers, the firms, and the 'public authority' acting in a relatively autonomous way". He has aptly called this class of equilibria, "politico-economic equilibria."

4.1. Every CCE is a Pareto Optimum.

We now prove the optimality of a collective consumption equilibrium.

Theorem 3: *Suppose u^h satisfies local nonsatiation for all $h \in H$ and $\langle p, c_g, \mathcal{T}, (x_o^h)_h, (y_o^i)_i, (z_o^j)_j, (x_n^h)_h, (y_n^i)_i, (z_n^j)_j, (z_g^j)_j\rangle$ is a CCE of a private ownership economy $E^P = \langle (X^h, u^h)_h, (Y^i)_i, (Z^j)_j, (\omega_o^h, \theta_y^h(i), \theta_z^h(j), \sigma_n^j)_{h,i,j}, \omega_n, \omega_g\rangle$ derived from economy E. Then the CCE is a Pareto optimum of economy E.*

Proof: Suppose $\langle p, c_g, \mathcal{T}, (x_o^h)_h, (y_o^i)_i, (z_o^j)_j, (x_n^h)_h, (y_n^i)_i, (z_n^j)_j, (z_g^j)_j\rangle$ is a CCE of E^P but it is not a Pareto optimum of E. Then there exists a feasible allocation $\overset{*}{a} := \langle (\overset{*}{x}^h)_h, (\overset{*}{y}^i)_i, (\overset{*}{z}^j)_j\rangle$ of E such that $\overset{*}{a}$ solves (4.4) with $\overset{*}{x}_g^h = \overset{*}{c}_g$, $\overset{*}{x}_n^h = \overset{*}{y}_n^i =: \overset{*}{n}$ for all h and i and $u^h(\overset{*}{x}_o^h, \overset{*}{c}_g, \overset{*}{x}_n^h) \geq u^h(x_o^h, c_g, x_n^h)$ for all h and $u^h(\overset{*}{x}_o^h, \overset{*}{c}_g, \overset{*}{x}_n^h) > u^h(x_o^h, c_g, x_n^h)$ for some h.[27] Pre-multiplying the second, third, and the last equalities in (4.4), evaluated at $\overset{*}{a}$, by p_o, p_{z_g}, and p_{z_n}, respectively, we obtain

$$\begin{aligned} &p_o \sum_h \overset{*}{x}_o^h + p_{z_g}\overset{*}{c}_g + p_{z_n}[\overset{*}{n} - \omega_n] \\ &= p_o \sum_i \overset{*}{y}_o^i + p_o \sum_j \overset{*}{y}_o^j + p_o\omega_o + p_{z_g}[\sum_j \overset{*}{z}_g^j + \omega_g] + p_{z_n} \sum_j \overset{*}{z}_n^j \\ &\leq \sum_i \hat{\Pi}_y^i(p_o, p_{y_n}^i, \overset{*}{c}_g) + \sum_j \Pi_z^j(p_o, p_{z_n}, p_{z_g}) - \sum_i p_{y_n}^i[\overset{*}{n} - \omega_n] + p_o\omega_o + p_{z_g}\omega_g. \end{aligned} \tag{4.5}$$

The last inequality is obtained by adding and subtracting $\sum_i p_{y_n}^i[\overset{*}{n} - \omega_n]$ from the second-last inequality. Hence, we have

$$\begin{aligned} p_o \sum_h \overset{*}{x}_o^h - \sum_i \hat{\Pi}_y^i(p_o, p_{y_n}^i, \overset{*}{c}_g) + \sum_h p_{x_n}^h \overset{*}{n} \leq \sum_j \Pi_z^j(p_o, p_{z_n}, p_{z_g}) - p_{z_g}[\overset{*}{c}_g - \omega_g] \\ + p_o\omega_o + \sum_h p_{x_n}^h \omega_n. \end{aligned} \tag{4.6}$$

On the other hand, local non-satiation of consumer preferences implies that the aggregate budget constraint of the consumers holds as an equality.[28]

$$\begin{aligned} p_o \sum_h x_o^h - \sum_i \hat{\Pi}_y^i(p_o, p_n^i, c_g) + \sum_h \mathcal{T}^h + \sum_h p_{x_n}^h n = \sum_j \Pi_z^j(p_o, p_{z_n}, p_{z_g}) \\ + p_o\omega_o + \sum_h p_{x_n}^h \omega_n. \end{aligned} \tag{4.7}$$

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[27] For convenience, we have assumed $\sigma_n = 0$.

[28] $n := x_n^h = y_n^i,\ \forall i \in I$ and $\forall h \in H$.

Subtracting (4.6) from (4.7), we obtain

$$\sum_h \mathcal{T}^h + p_o \sum_h [x_o^h - \overset{*}{x}{}_o^h] - \sum_i [\hat{\Pi}_y^i(p_o, p_{y_n}^i, c_g) - \hat{\Pi}_y^i(p_o, p_{y_n}^i, \overset{*}{c}_g)] + \sum_h p_{x_n}^h [n - \overset{*}{n}] \geq p_{z_g}[\overset{*}{c}_g - \omega_g]. \tag{4.8}$$

For all h, choose

$$\overset{*}{\mathcal{T}}{}^h = \mathcal{T}^h + p_o[x_o^h - \overset{*}{x}{}_o^h] - \sum_i \theta_y^h(i)[\hat{\Pi}_y^i(p_o, p_{y_n}^i, c_g) - \hat{\Pi}_y^i(p_o, p_{y_n}^i, \overset{*}{c}_g)] + p_{x_n}^h[n - \overset{*}{n}]. \tag{4.9}$$

Then, it follows from (4.8) and (4.9) that

$$\sum_h \overset{*}{\mathcal{T}}{}^h \geq p_{z_g}[\overset{*}{c}_g - \omega_g]. \tag{4.10}$$

Further, from (4.9) and consumer budget balance, we have

$$\begin{aligned} &\overset{*}{\mathcal{T}}{}^h + p_o \overset{*}{x}{}_o^h - \sum_i \theta_y^h(i)\hat{\Pi}_y^i(p_o, p_{y_n}^i, \overset{*}{c}_g) + p_{x_n}^h \overset{*}{n} \\ &= \mathcal{T}^h + p_o x_o^h - \sum_i \theta_y^h(i)\hat{\Pi}_y^i(p_o, p_{y_n}^i, c_g) + p_{x_n}^h n \\ &= \sum_j \theta_z^h(j)\Pi_z^j(p_o, p_d, p_z) + p_o\omega^h + p_{x_n}^h \eta. \end{aligned} \tag{4.11}$$

This implies that, for all $h \in H$, we have

$$p_o \overset{*}{x}{}_o^h + p_{x_n}^h \overset{*}{n} = \sum_i \theta_y^h(i)\hat{\Pi}_y^i(p_o, p_{x_n}^i, \overset{*}{c}_g) + \sum_j \theta_z^h(j)\Pi^j(p_o, p_{z_n}, p_{z_g}) + p_o\omega_o^h + p_{x_n}^h\omega_n - \overset{*}{\mathcal{T}}{}^h. \tag{4.12}$$

Thus, the bundle $\langle \overset{*}{x}{}_o^h, \overset{*}{n}{}^h \rangle$ is affordable with income $r^h(p, \overset{*}{\mathcal{T}}{}^h, \overset{*}{c}_g)$ for all $h \in H$. Now define A such that

$$\sum_h \overset{*}{\mathcal{T}}{}^h = A + p_{z_g}[\overset{*}{c}_g - \omega_g]. \tag{4.13}$$

From (4.10), we have $A \geq 0$. Define, for all $h \in H$,

$$\hat{\mathcal{T}}^h = \overset{*}{\mathcal{T}}{}^h - \frac{A}{H}. \tag{4.14}$$

Then, for all $h \in H$, we have

$$r^h(p, \hat{\mathcal{T}}^h, \overset{*}{c}_g) = r^h(p, \overset{*}{\mathcal{T}}{}^h, \overset{*}{c}_g) + \frac{A}{H} \geq r^h(p, \overset{*}{\mathcal{T}}{}^h, \overset{*}{c}_g), \tag{4.15}$$

$$V^h(p, \overset{*}{c}_g, r^h(p, \hat{\mathcal{T}}^h, \overset{*}{c}_g) \geq V^h(p, \overset{*}{c}_g, r^h(p, \overset{*}{\mathcal{T}}{}^h, \overset{*}{c}_g) \geq u^h(\overset{*}{x}{}_o^h, \overset{*}{x}{}_n^h, \overset{*}{c}_g) \geq u^h(x_o^h, x_n^h, c_g), \tag{4.16}$$

$$\sum_h \hat{\mathcal{T}}^h = p_{z_g}[\overset{*}{c}_g - \omega_g], \text{ and for some } h \tag{4.17}$$

$$V^h(p, \overset{*}{c}_g, r^h(p, \hat{\mathcal{T}}^h, \overset{*}{c}_g) \geq V^h(p, \overset{*}{c}_g, r^h(p, \overset{*}{\mathcal{T}}^h, \overset{*}{c}_g) \geq u^h(\overset{*}{x}{}^h_o, \overset{*}{x}{}^h_n, \overset{*}{c}_g) > u^h(x^h_o, x^h_n, c_g), \tag{4.18}$$

Thus, we have created a government budget proposal, $\langle \hat{\mathcal{T}}, \overset{*}{c}_g \rangle$ relative to price system p such that the proposal $\langle \mathcal{T}, c_g \rangle$ will be unanimously rejected in favor of $\langle \hat{\mathcal{T}}, \overset{*}{c}_g \rangle$. This contradicts the fact that $\langle p, c_g, \mathcal{T}, (x^h_o)_h, (y^i_o)_i, (z^j_o)_j, (x^h_n)_h, (y^i_n)_i, (z^j_n)_j, (z^j_g)_j \rangle$ is a CCE of E^P.■

At a CCE, the lump-sum taxes in a government proposal incorporate both an element of willingness to pay for the level of public commodities specified in the proposal and an element of redistribution. To the extent there are no restrictions on the redistributive powers of the collective action and because a CCE is Pareto optimal, given any private ownership economy, the set of CCE allocations associated with it is the entire Pareto frontier.[29] The choice of the collective action can be narrowed if it adopts a specific political mechanism. There are many forms such a mechanism might take. In the interest of equity, Foley [1967] explores the existence of CCE where the tax structure is progressive. Alternatively, the collective action can have a utilitarian social welfare function as its objective function for determining the optimal government proposal at prevailing market prices. Another motivation for narrowing the choice of the collective action could be incentive constraints. It is interesting to note that the feasible sets of some of the standard incentive compatible mechanisms for public commodities, such as the Clarke-Groves mechanism, take the form of government proposals, as defined above. In fact, for quasi-linear preferences, at every price vector, the collective action can run a Clarke-Groves mechanism which will induce the victims to reveal truthful preferences. This determines the demand of the collective action at those prices. The demand interacts with the supply and an equilibrium price configuration is obtained.

4.2. Decentralization of a Pareto Optimum as a Restricted Collective Consumption Equilibrium.

Because of the technological nonconvexities that can arise for firms in I, we find that a given Pareto optimum of E is only a *restricted* collective consumption equilibrium, that is, it is a collective consumption equilibrium of a restricted economy where the set of government budget proposals, from which the consumers vote, is restricted to those for which the levels of the public commodities are fixed (in this case, at the Pareto optimal levels). So consumers accept or reject based on the distribution of the contributions alone.

Definition. A *restricted collective consumption equilibrium (RCCE)* of $E^P = \langle (X^h, u^h)_h, (Y^i)_i, (Z^j)_j, (\omega^h_o, \theta^h_y(i), \theta^h_z(j), \sigma^j_n)_{h,i,j}, \omega_n, \omega_g \rangle$ is a configuration $\langle p, c_g, T, (x^h_o)_h, (y^i_o)_i, (z^j_o)_j, (x^h_n)_h, (y^i_n)_i, (z^j_n)_j, (z^j_g)_j \rangle$ such that (i) $\langle (x^h_o, x^h_n)_h, (y^i_o, y^i_n)_i, (z^j_o, z^j_n, z^j_g)_j \rangle$ solve (4.3), (3.2), and (3.4), respectively, for $\langle p, c_g, T \rangle$; (ii) $\langle c_g, T \rangle$ is a government budget proposal relative to price system p that is not unanimously rejected in favor of any other government budget proposal $\langle \bar{c}_g, \bar{T} \rangle$ relative to price system p with $\bar{c}_g = c_g$; and (iii) (4.4) holds.

The proof of the second welfare theorem version for RCCE (Theorem 4) will employ a lemma. This lemma uses the Clarke's normal cone (the negative polar cone of the Clarke's tangent cone) to identify the cones of shadow prices associated with a Pareto optimal alloca-

29 See Foley [1967] p. 57 and Milleron pp. 432, 433, 436.

tion.[30] The Clarke's tangent cone (normal cone) for $Y \subset \mathbf{R}^n$ relative to $y \in Cl\ Y$ is denoted by $T(Y, y)$ $(N(Y, y))$.[31]

Lemma 1: *Let* $\langle(\overset{*}{x}^h)_h, (\overset{*}{y}^i)_i, (\overset{*}{z}^j)_j\rangle$ *be a Pareto optimal state of* E. *Let* $\overset{*}{u}^h = u^h(\overset{*}{x}^h)$ *for all* h. *Suppose* Y^i *and* Z^j *are closed for all* i *and* j, u^h *is continuous for all* h, *and the following hold: (a)* $int\ T(R^h(\overset{*}{u}^h), \overset{*}{x}^h)) \neq \emptyset$ *for all* h, *(b)* $int\ T(Y^i, \overset{*}{y}^i) \neq \emptyset$ *for all* i, *(c)* $int\ T(Z^j, \overset{*}{z}^j) \neq \emptyset$ *for all* j, *and (d) there exists* $\bar{h}$ *for whom* $R^{\bar{h}}(\overset{*}{u}^{\bar{h}}) \subseteq Cl\ P^{\bar{h}}(\overset{*}{u}^{\bar{h}})$ *(existence of a locally nonsatiated consumer). Then for all* h, i, *and* j, *there exists* $-\overset{*}{\rho}{}_x^h \in N(R^h(\overset{*}{u}^h), \overset{*}{x}^h)$, $\overset{*}{\rho}{}_y^i \in N(Y^i, \overset{*}{y}^i)$, *and* $\overset{*}{\rho}{}_z^j \in N(Z^j, \overset{*}{z}^j)$, *not all equal to* 0, *such that (1)* $-\overset{*}{\rho}{}_{x_o}^h = \overset{*}{\rho}{}_{y_o}^i = \overset{*}{\rho}{}_{z_o}^j$, *(2)* $\overset{*}{\rho}{}_{z_g}^j = \sum_h \overset{*}{\rho}{}_{x_g}^h - \sum_i \overset{*}{\rho}{}_{y_g}^i =: \overset{*}{\rho}_g$ *and (3)* $\overset{*}{\rho}{}_{z_n}^j = \sum_h \overset{*}{\rho}{}_{x_n}^h - \sum_i \overset{*}{\rho}{}_{y_n}^i =: \overset{*}{\rho}_n$.

Proof: Let $\mathcal{A} = \mathbf{R}^{(H+I+J)(O+G+N)}$. A typical vector in $\mathcal{A}$ is denoted by $s = \langle(x^h)_h, (y^i)_i, (z^j)_j\rangle$. We define the embedding of $R^h(\overset{*}{u}^h)$ in $\mathcal{A}$ as $\widehat{R}^h(\overset{*}{u}^h) := \{s \in \mathcal{A} | x^h \in R^h(\overset{*}{u}^h)\}$. Similarly, we define $\widehat{P}^h(\overset{*}{u}^h)$, $\widehat{Y}^i$, and $\widehat{Z}^j$ for all h, i, and j. Let $\mathcal{W}_o(\alpha) := \{s \in \mathcal{A} | \sum_h x_o^h(\alpha) \leq \sum_i y_o^i(\alpha) + \sum_j z_o^j(\alpha) + \omega_o(\alpha)\}$ for all α; $\mathcal{W}_{x_g}^h(\gamma) := \{s \in \mathcal{A} | x_g^h(\gamma) - \omega_g(\gamma) \leq \sum_j z_g^j(\gamma)\}$ for all h and γ; $\mathcal{W}_{y_g}^i(\gamma) := \{s \in \mathcal{A} | y_g^i(\gamma) - \omega_g(\gamma) \leq \sum_j z_g^j(\gamma)\}$ for all i and γ; $\mathcal{W}_{x_n}^h(\beta) := \{s \in \mathcal{A} | x_n^h(\beta) - \omega_n(\beta) \leq \sum_j z_n^j(\beta) + \sigma_n(\beta)\}$ for all h and $\beta \in N$; and $\mathcal{W}_{y_n}^i(\beta) := \{s \in \mathcal{A} | y_n^i(\beta) - \omega_n(\beta) \leq \sum_j z_n^j(\beta) + \sigma_n(\beta)\}$ for all i and $\beta \in N$.
Let $\overset{*}{s} := \langle(\overset{*}{x}^h)_h, (\overset{*}{y}^i)_i, (\overset{*}{z}^j)_j\rangle$. It follows from Lemma A.5 and the maintained Assumptions (a) to (c) that $int\ T(\widehat{R}^h(\overset{*}{u}^h), \overset{*}{s})$, $int\ T(\widehat{Y}^i, \overset{*}{s})$, and $int\ T(\widehat{Z}^j, \overset{*}{s})$ are non empty for all h, i, and j. From Lemma A.2 it follows that $int\ T(\mathcal{W}_o(\alpha), \overset{*}{s})$, $int\ T(\mathcal{W}_{x_g}^h(\gamma), \overset{*}{s})$, $int\ T(\mathcal{W}_{y_g}^i(\gamma), \overset{*}{s})$, $int\ T(\mathcal{W}_{x_n}^h(\beta), \overset{*}{s})$, and $int\ T(\mathcal{W}_{y_n}^i(\beta), \overset{*}{s})$ are not empty for all α, γ, β, h, i, and j. Next, note that since $\overset{*}{s}$ corresponds to a Pareto optimum, we have

$$\begin{aligned} V := &\bigcap_{h \neq \bar{h}} \widehat{R}^h(\overset{*}{u}^h) \bigcap \widehat{P}^{\bar{h}}(\overset{*}{u}^{\bar{h}}) \bigcap_i \widehat{Y}^i \bigcap_j \widehat{Z}^j \bigcap_\alpha \mathcal{W}_o(\alpha) \bigcap_{i,\gamma} \mathcal{W}_{y_g}^i(\gamma) \bigcap_{i,\beta} \mathcal{W}_{y_n}^i(\beta) \\ &\bigcap_{h,\gamma} \mathcal{W}_{x_g}^h(\gamma) \bigcap_{h,\beta} \mathcal{W}_{x_n}^h(\beta) = \emptyset. \end{aligned} \tag{4.19}$$

We show that this implies

$$\begin{aligned} &\hat{V} := \bigcap_h int\ T(\widehat{R}^h(\overset{*}{u}^h), \overset{*}{s}) \bigcap_i int\ T(\widehat{Y}^i, \overset{*}{s}) \bigcap_j int\ T(\widehat{Z}^j, \overset{*}{s}) \bigcap_\alpha int\ T(\mathcal{W}_o(\alpha), \overset{*}{s}) \\ &\bigcap_{i,\gamma} int\ T(\mathcal{W}_{y_g}^i(\gamma), \overset{*}{s}) \bigcap_{i,\beta} int\ T(\mathcal{W}_{y_n}^i(\beta), \overset{*}{s}) \bigcap_{h,\gamma} int\ T(\mathcal{W}_{x_g}^h(\gamma), \overset{*}{s}) \bigcap_{h,\beta} int\ T(\mathcal{W}_{x_n}^h(\beta), \overset{*}{s}) = \emptyset. \end{aligned} \tag{4.20}$$

[30] This conceptualization of shadow prices and other alternative conceptualizations have been extensively used in the literature on nonconvex economies. See, for instance, Beato and Mas-Colell [1985], Bonniseau and Cornet [1990], Brown and Heal [1979], Guesnerie (1975), Khan and Vohra [1987], and Quinzii [1992].

[31] We denote the closure of a set $Y \in \mathbf{R}^n$ by $Cl\ Y$ and the interior of Y relative to $\mathbf{R}^n$ by $int\ Y$. For definitions of these cones and their properties, see Clarke [1975, 1983, 1989], Rockafellar [1978], Khan and Vohra [1987], and Cornet [1989]. The properties of the Clarke's tangent and normal cones that are relevant for this paper can be found in the appendix in the form of lemmas.

Define

$$\bar{V} := \bigcap_{h\neq\bar{h}} \widehat{R}^h(\overset{*}{u}^h) \bigcap_{\bar{h}} Cl\ \widehat{P}^{\bar{h}}(\overset{*}{u}^{\bar{h}}) \bigcap_i \widehat{Y}^i \bigcap_j \widehat{Z}^j \bigcap_\alpha \mathcal{W}_o(\alpha) \bigcap_{i,\gamma} \mathcal{W}^i_{y_g}(\gamma) \bigcap_{i,\beta} \mathcal{W}^i_{y_n}(\beta) \bigcap_{h,\gamma} \mathcal{W}^h_{x_g}(\gamma) \bigcap_{h,\beta} \mathcal{W}^h_{x_n}(\beta). \tag{4.21}$$

Note that, because of Assumption (d), $\overset{*}{s} \in \bar{V}$. We show that $int\ T(\bar{V}, \overset{*}{s}) = \emptyset$. For, from this and from Lemma A.3 and the maintained Assumption (d) it will follow that $\hat{V} = \emptyset$. Suppose $v \in int\ T(\bar{V}, \overset{*}{s})$. Then using the definition of the interior of the tangent cone, there exist $\epsilon > 0$ and $\eta > 0$ such that $\{\overset{*}{s}\} + \lambda Cl\ \mathcal{N}_\epsilon(v) \subset \bar{V} \subset Cl\ \widehat{P}^{\bar{h}}(\overset{*}{u}^{\bar{h}})$ for all $\lambda \in [0, \eta]$. Pick $a \in \{\overset{*}{s}\} + \lambda \mathcal{N}_\epsilon(v)$ for a $\lambda \in [0, \eta]$. Then $a \in Cl\ \widehat{P}^{\bar{h}}(\overset{*}{u}^{\bar{h}})$. Since a is a limit point of $\widehat{P}^{\bar{h}}(\overset{*}{u}^{\bar{h}})$, for any $\delta > 0$ such that $\mathcal{N}_\delta(a) \subseteq \{\overset{*}{s}\} + \lambda \mathcal{N}_\epsilon(v)$, we have $\mathcal{N}_\delta(a) \cap \widehat{P}^{\bar{h}}(\overset{*}{u}^{\bar{h}}) \neq \emptyset$. Pick $b \in \mathcal{N}_\delta(a) \cap \widehat{P}^{\bar{h}}(\overset{*}{u}^{\bar{h}}) \subseteq \{\overset{*}{s}\} + \lambda \mathcal{N}_\epsilon(v) \subset \bar{V}$. Thus $b \in V$. This is a contradiction to (4.19), that is, a contradiction to $\overset{*}{s}$ being a Pareto optimal state. Hence $int\ T(\bar{V}, \overset{*}{s}) = \emptyset$.
By Lemma A.4, $\exists\ -\overset{*}{\sigma}^h_x \in N(\widehat{R}^h(\overset{*}{u}^h), \overset{*}{s})$, $\overset{*}{\sigma}^i_y \in N(\widehat{Y}^i, \overset{*}{s})$, $\overset{*}{\sigma}^j_z \in N(\widehat{Z}^j, \overset{*}{s})$, $\psi^\alpha_o \in N(\mathcal{W}_o(\alpha), \overset{*}{s})$, $\psi^h_{x_g(\gamma)} \in N(\mathcal{W}^h_{x_g}(\gamma), \overset{*}{s})$, $\psi^h_{x_n(\beta)} \in N(\mathcal{W}^h_{x_n}(\beta), \overset{*}{s})$, $\psi^i_{y_g(\gamma)} \in N(\mathcal{W}^i_{y_g}(\gamma), \overset{*}{s})$, and $\psi^i_{y_n(\beta)} \in N(\mathcal{W}^i_{y_n}(\beta), \overset{*}{s})$ for all α, h, γ, β, and i, not all equal to 0, such that

$$\sum_h \overset{*}{\sigma}^h_x + \sum_i \overset{*}{\sigma}^i_y + \sum_j \overset{*}{\sigma}^j_z + \sum_\alpha \psi^\alpha_o + \sum_{h,\gamma} \psi^h_{x_g(\gamma)} + \sum_{h,\beta} \psi^h_{x_n(\beta)} + \sum_{i,\gamma} \psi^i_{y_g(\gamma)} + \sum_{i,\beta} \psi^i_{y_n(\beta)} = 0. \tag{4.22}$$

Employing Lemmas A.2 and A.5 and working through element-by-element of the left-hand side of (4.22), we find that, for all h, i, j, α, β, and γ, there exist $-\overset{*}{\rho}^h_x \in N(R^h(\overset{*}{u}^h), \overset{*}{s})$, $\overset{*}{\rho}^i_y \in N(Y^i, \overset{*}{s})$, and $\overset{*}{\rho}^j_z \in N(Z^j, \overset{*}{s})$ and scalars $\lambda_\alpha \geq 0$, $\lambda^i_{y_g(\gamma)} \geq 0$, $\lambda^i_{y_n(\beta)} \geq 0$, $\lambda^h_{x_g(\gamma)} \geq 0$, and $\lambda^h_{x_n(\beta)} \geq 0$ such that the following are true:
(1′) $\overset{*}{\rho}^h_{x_o}(\alpha) = \overset{*}{\rho}^i_{y_o}(\alpha) = \overset{*}{\rho}^j_{z_o}(\alpha) = \lambda_\alpha$, (2′) $\overset{*}{\rho}^h_{x_g}(\gamma) = \lambda^h_{x_g(\gamma)}$, (3′) $\overset{*}{\rho}^h_{x_n}(\beta) = \lambda^h_{x_n(\beta)}$, (4′) $\overset{*}{\rho}^i_{y_g}(\gamma) = -\lambda^i_{y_g(\gamma)}$, (5′) $\overset{*}{\rho}^i_{y_n}(\beta) = -\lambda^i_{y_n(\beta)}$, (6′) $\overset{*}{\rho}^j_{z_g}(\gamma) = \sum_h \lambda^h_{x_g(\gamma)} - \sum_i \lambda^i_{y_g(\gamma)}$, and (7′) $\overset{*}{\rho}^j_{z_n}(\beta) = \sum_h \lambda^h_{x_n(\beta)} - \sum_i \lambda^i_{y_n(\beta)}$.[32] Conclusions of Lemma 1 follow.■

Theorem 4: *Suppose $\langle(\overset{*}{x}^h)_h, (\overset{*}{y}^i)_i, (\overset{*}{z}^j)_j\rangle$ is a Pareto optimum of E and the following assumptions hold:*
(i) $\overset{*}{x}^h \in int\ X^h$, for all $h \in H$,[33]
(ii) u^h is quasi-concave, locally nonsatiated, and continuous, for all $h \in H$,
(iii) Z^j is convex, closed, and $int\ Z^j \neq \emptyset$ for all $j \in J$, and
(iv) Y^i is closed, $int\ T(Y^i, \overset{*}{y}^i) \neq \emptyset$, and $Y^i(y^i_g)$ is convex for all $y^i_g \in \mathbf{R}^G_+$ and for all $i \in I$,
Then for every distribution of initial resources (including collective ownership of ω_g and ω_n) and systems of shares $\langle(\omega^h_o)_h, (\theta^h_y(i))_{h,i}, (\theta^h_z(j))_{h,j}, (\sigma^j_n)_j\rangle$, there exists a price system $\overset{*}{p}$ and taxes $\overset{*}{T}$ such that the implied configuration $\langle\overset{*}{p}, \overset{*}{c}_g, \overset{*}{T}, (\overset{*}{x}^h_o)_h, (\overset{*}{y}^i_o)_i, (\overset{*}{z}^j_o)_j, (\overset{*}{x}^h_n)_h, (\overset{*}{y}^i_n)_i, (\overset{*}{z}^j_n)_j, (\overset{*}{z}^j_g)_j\rangle$ is a RCCE of $E^P = \langle(X^h, u^h)_h, (Y^i)_i, (Z^j)_j, (\omega^h_o, \theta^h_y(i), \theta^h_z(j), \sigma^j_n)_{h,i,j}, \omega_n, \omega_g\rangle$ derived from economy E.

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32 For details, see the appendix.
33 An alternate assumption, called *irreducibility*, can be found in Ghosal and Polemarchakis [1999].

Proof: Assumptions (ii), (iii), and (iv) in the theorem along with Lemma A.6 ensure that Assumptions (a) to (d) of Lemma 1 hold. Hence, conclusions of Lemma 1 follow, and let us define $\overset{*}{p}_o = \overset{*}{\rho}_o$, $\overset{*}{p}_{z_g} = \overset{*}{\rho}_g$, $\overset{*}{p}^h_{x_n} = \overset{*}{\rho}^h_{x_n}$, for all $h \in H$, $\overset{*}{p}^i_{y_n} = \overset{*}{\rho}^i_{y_n}$, for all $i \in I$, and $\overset{*}{p}_{z_n} = \overset{*}{\rho}_n$.
Pick any distribution of endowments and profit shares $\langle(\omega^h_o)_h, (\theta^h_y(i))_{h,i}, (\theta^h_z(j))_{h,j}, (\sigma^j_N)_j\rangle$. Since $\overset{*}{\rho}^j_z \in N(Z^j, \overset{*}{z}^j)$ for all $j \in J$ and Z^j is convex, from Lemma A.7 it follows that $\Pi^j_z(\overset{*}{p}_o, \overset{*}{p}_{z_n}, \overset{*}{p}_{z_g}) = \overset{*}{p}_o \cdot \overset{*}{z}^j_o + \overset{*}{p}_{z_n} \cdot [\overset{*}{z}^j_n + \sigma^j_n] + \overset{*}{p}_{z_g} \cdot \overset{*}{z}^j_g$. Since $\overset{*}{\rho}^i \in N(Y^i, \overset{*}{y}^i)$ and $Y^i(\overset{*}{c}_g)$ is convex for all $i \in I$, it follows from Lemma A.7 that $\hat{\Pi}^i_y(\overset{*}{p}_o, \overset{*}{p}^i_{y_n}, \overset{*}{c}_g) = \overset{*}{p}_o \cdot \overset{*}{y}^i_o + \overset{*}{p}^i_{y_n} \cdot [\overset{*}{y}^i_n - \omega_n]$. For all $h \in H$, we define $\overset{*}{\mathcal{T}}^h$, $\overset{*}{A}^h$, and $\overset{*}{r}^h$ as

$$
\begin{aligned}
-\overset{*}{\mathcal{T}}^h &= \overset{*}{p}_o \overset{*}{x}^h_o + \overset{*}{p}^h_{x_n} \overset{*}{x}^h_n - \sum_i \theta^h_y(i) \hat{\Pi}^i_y(\overset{*}{p}_o, \overset{*}{p}^i_{y_n}, \overset{*}{c}_g) - \overset{*}{A}^h \\
\overset{*}{A}^h &= \sum_j \theta^h_z(j) \Pi^j_z(\overset{*}{p}_o, \overset{*}{p}_{z_n}, \overset{*}{p}_{z_g}) + \overset{*}{p}_o \omega^h_o + \overset{*}{p}^h_{x_n} \omega_n \\
\overset{*}{r}^h &:= r^h(\overset{*}{p}^h, \overset{*}{c}_g, \overset{*}{\mathcal{T}}^h) = -\overset{*}{\mathcal{T}}^h + \overset{*}{A}^h + \sum_i \theta^h_y(i) \hat{\Pi}^i_y(\overset{*}{p}_o, \overset{*}{p}^i_{y_n}, \overset{*}{c}_g) = \overset{*}{p}_o \overset{*}{x}^h_o + \overset{*}{p}^h_{x_n} \overset{*}{x}_n.
\end{aligned}
\tag{4.23}
$$

For all $h \in H$ let $u^h(\overset{*}{x}^h) = \overset{*}{u}^h$. Since $-\overset{*}{\rho}^h_x \in N(R^h(\overset{*}{u}^h), \overset{*}{x}^h)$, X^h is convex, u^h satisfies local nonsatiation and is continuous and quasiconcave for all $h \in H$, by employing Lemma A.7 we obtain (1) $R^h(\overset{*}{u}^h)$ is convex, (2) $\overset{*}{x}^h$ is a boundary point of $R^h(\overset{*}{u}^h)$, (3) $\overset{*}{\rho}^h_x \cdot \overset{*}{x}^h \leq \overset{*}{\rho}^h_x \cdot x^h$ for all $x^h \in R^h(\overset{*}{u}^h)$ ($\overset{*}{x}^h$ is cost minimizing in $R^h(\overset{*}{u}^h)$), (4) $R^h(\overset{*}{u}^h, \overset{*}{c}_g)$ is also convex, (5) $-\langle\overset{*}{\rho}^h_{x_o}, \overset{*}{\rho}^h_{x_n}\rangle \in N(R^h(\overset{*}{u}^h, \overset{*}{c}_g), \langle\overset{*}{x}^h_o, \overset{*}{x}^h_n\rangle)$, and (6) $\overset{*}{\rho}^h_{x_o} \cdot \overset{*}{x}^h_o + \overset{*}{\rho}^h_{x_n} \cdot \overset{*}{x}^h_n \leq \overset{*}{\rho}^h_{x_o} \cdot x^h_o + \overset{*}{\rho}^h_{x_n} \cdot x^h_n$ for all $\langle x^h_o, n^h_o\rangle \in R^h(\overset{*}{u}^h, \overset{*}{c}_g)$.[34] In addition, since $\overset{*}{x}^h \in int\ X^h$ for all $h \in H$, we have $\langle\overset{*}{x}^h_o, \overset{*}{x}^h_n\rangle \in int\{\langle x^h_o, x^h_n\rangle |\ \langle x^h_o, x^h_n, \overset{*}{c}_g\rangle \in X^h\}$. Hence, for all $h \in H$, we have $u^h(\overset{*}{x}^h_o, \overset{*}{x}^h_n, \overset{*}{c}_g) \geq u^h(x^h_o, x^h_n, \overset{*}{c}_g)$ for all $\langle x^h_o, x^h_n\rangle \in \{\langle x^h_o, x^h_n\rangle |\ \langle x^h_o, x^h_n, \overset{*}{c}_g\rangle \in X^h$ such that $\overset{*}{\rho}^h_{x_o} \cdot x^h_o + \overset{*}{\rho}^h_{x_n} \cdot x^h_n \leq \overset{*}{\rho}^h_{x_o} \cdot \overset{*}{x}^h_o + \overset{*}{\rho}^h_{x_n} \cdot \overset{*}{x}^h_n = \overset{*}{r}^h\}$ (the last equality follows from (4.23)).[35] Thus, $V^h(\overset{*}{p}_o, \overset{*}{p}^h_{x_n}, \overset{*}{c}_g, \overset{*}{r}^h) = \overset{*}{u}^h$ for all $h \in H$.
Thus, the configuration $\langle\overset{*}{p}, \overset{*}{c}_g, \overset{*}{\mathcal{T}}, (\overset{*}{x}^h_o)_h, (\overset{*}{y}^i_o)_i, (\overset{*}{z}^j_o)_j, (\overset{*}{x}^h_n)_h, (\overset{*}{y}^i_n)_i, (\overset{*}{z}^j_n)_j, (\overset{*}{z}^j_g)_j\rangle$ satisfies parts (i) and (iii) of the definition of a RCCE of E^P. We now show that part (ii) of this definition is also satisfied.
First note, local nonsatiation of u^h for all $h \in H$ (which implies that consumer budget constraints hold as equalities), feasibility of the Pareto allocation, and the fact that $\overset{*}{p}_{z_n} = \sum_h \overset{*}{p}^h_{x_n} - \sum_i \overset{*}{p}^i_{y_n}$ at the Pareto optimal allocation (this follows from Lemma 1) imply, as a consequence of the Walras law, that $\sum_h \overset{*}{\mathcal{T}}^h + \overset{*}{p}_{z_g}[\overset{*}{c}_g - \omega_g] = 0$. Suppose, there existed a government budget proposal $\langle\overset{*}{c}_g, \mathcal{T}\rangle$ relative to price system $\overset{*}{p}$ such that the government budget proposal $\langle\overset{*}{c}_g, \overset{*}{\mathcal{T}}\rangle$ is unanimously rejected in favor of government budget proposal $\langle\overset{*}{c}_g, \mathcal{T}\rangle$. Let, for all $h \in H$, $\langle x^h_o, x^h_n\rangle$ solve (4.3) for price system $\overset{*}{p}$ and income $r^h = r^h(\overset{*}{p}, \mathcal{T}^h, \overset{*}{c}_g)$. Since, $\langle\overset{*}{x}^h_o, \overset{*}{x}^h_n\rangle$ solves (4.3) for price system $\overset{*}{p}$ and income $\overset{*}{r}^h = r^h(\overset{*}{p}, \overset{*}{\mathcal{T}}^h, \overset{*}{c}_g)$ and local nonsatiation is true, we have

$$
\begin{aligned}
\overset{*}{\rho}^h_{x_o} \cdot x^h_o + \overset{*}{\rho}^h_{x_n} \cdot x^h_n &= \sum_i \theta^h_y(i) \hat{\Pi}^i_y(\overset{*}{p}_o, \overset{*}{p}^i_{y_n}, \overset{*}{c}_g) - \overset{*}{A}^h - \mathcal{T}^h \text{ and} \\
\overset{*}{\rho}^h_{x_o} \cdot \overset{*}{x}^h_o + \overset{*}{\rho}^h_{x_n} \cdot \overset{*}{x}^h_n &= \sum_i \theta^h_y(i) \hat{\Pi}^i_y(\overset{*}{p}_o, \overset{*}{p}^i_{y_n}, \overset{*}{c}_g) - \overset{*}{A}^h - \overset{*}{\mathcal{T}}^h.
\end{aligned}
\tag{4.24}
$$

[34] See Debreu [1959].

[35] See Debreu [1959].

Hence, for all $h \in H$, we have

$$\overset{*}{\rho}{}^h_{x_o} \cdot x^h_o + \overset{*}{\rho}{}^h_{x_n} \cdot x^h_n + \mathcal{T}^h = \overset{*}{\rho}{}^h_{x_o} \cdot \overset{*}{x}{}^h_o + \overset{*}{\rho}{}^h_{x_n} \cdot \overset{*}{x}{}^h_n + \overset{*}{\mathcal{T}}{}^h. \tag{4.25}$$

Summing up over all h and recalling that $\sum_h \mathcal{T}^h = \overset{*}{p}_{z_g} \overset{*}{c}_g = \sum_h \overset{*}{\mathcal{T}}{}^h$, we have

$$\sum_h \overset{*}{\rho}{}^h_{x_o} \cdot x^h_o + \sum_h \overset{*}{\rho}{}^h_{x_n} \cdot x^h_n + \overset{*}{p}_{z_g} \overset{*}{c}_g = \sum_h \overset{*}{\rho}{}^h_{x_o} \cdot \overset{*}{x}{}^h_o + \sum_h \overset{*}{\rho}{}^h_{x_n} \cdot \overset{*}{x}{}^h_n + \overset{*}{p}_{z_g} \overset{*}{c}_g. \tag{4.26}$$

This implies, from Lemma 1 and our definition of $\overset{*}{p}_{z_g}$, that

$$\begin{aligned} \sum_h \overset{*}{\rho}{}^h_{x_o} \cdot x^h_o + \sum_h \overset{*}{\rho}{}^h_{x_n} \cdot x^h_n + [\sum_h \overset{*}{\rho}{}^h_{x_g} - \sum_i \overset{*}{\rho}{}^i_{y_g}] \overset{*}{c}_g = \sum_h \overset{*}{\rho}{}^h_{x_o} \cdot \overset{*}{x}{}^h_o + \sum_h \overset{*}{\rho}{}^h_{x_n} \cdot \overset{*}{x}{}^h_n \\ + [\sum_h \overset{*}{\rho}{}^h_{x_g} - \sum_i \overset{*}{\rho}{}^i_{y_g}] \overset{*}{c}_g. \end{aligned} \tag{4.27}$$

Hence, we have

$$\sum_h [\overset{*}{\rho}{}^h_{x_o} \cdot x^h_o + \overset{*}{\rho}{}^h_{x_n} \cdot x^h_n + \overset{*}{\rho}{}^h_{x_g} \overset{*}{c}_g] = \sum_h [\overset{*}{\rho}{}^h_{x_o} \cdot \overset{*}{x}{}^h_o + \overset{*}{\rho}{}^h_{x_n} \cdot \overset{*}{x}{}^h_n + \overset{*}{\rho}{}^h_{x_g} \overset{*}{c}_g] = \overset{*}{x}{}^h \cdot \overset{*}{\rho}{}^h_x. \tag{4.28}$$

Since, $\langle x^h_o, x^h_n, \overset{*}{c}_g \rangle \in P^h(\overset{*}{u}{}^h)$ and u^h is continuous, we have $\langle x^h_o, x^h_n, \overset{*}{c}_g \rangle \in int\ P^h(\overset{*}{u}{}^h)$ and there exists $\bar{x} := \langle \bar{x}^h_o, \bar{x}^h_o, \overset{*}{c}_g \rangle \in P^h(\overset{*}{u}{}^h)$ such that $\bar{x}^h \cdot \overset{*}{\rho}{}^h_x < \overset{*}{x}{}^h \cdot \overset{*}{\rho}{}^h_x$. This contradicts the fact that, for all $h \in H$, $\overset{*}{x}{}^h$ is cost minimizing in $R^h(\overset{*}{u}{}^h)$ at shadow price vector $\overset{*}{\rho}{}^h_x$.■

5. Conclusions.

In this paper, we reconcile Arrow/Starrett and Boyd and Conley general equilibrium models of externality. We argue that the Boyd and Conley externalities arise as a result of missing markets for certain nonproducible goods, which have public-commodity properties for some agents. Boyd and Conley's assumption that these resources are bounded and have positive values for all agents, alleviates the externality problem once Coasian markets are created for these goods. Arrow/Starrett externalities, on the other hand, have been argued as arising because of missing markets for certain producible goods. To the extent their disposal can be costly for the firms producing them and they can be detrimental and, hence, sources of technological nonconvexities for the victim firms, the externality problem for such goods cannot be solved by creating markets. The nonconvexities that they induce further implies that Hurwicz's impossibility result for finding *efficient* finite dimensional decentralized mechanisms applies. In this paper we propose an efficient partially decentralized mechanism motivated by the concept of a "public competitive equilibrium" defined by Foley. Precisely because his mechanism combines price and quantity signals and allows decentralized choice based on a unanimity criterion, it permits a concept of an equilibrium for general (including nonconvex) economies that will always be efficient. This is in contrast to equilibrium concepts based on Pigovian taxes.

Furthermore, for unilateral external effects, the informational requirements for the design of Foley type mechanism are less stringent (requiring the collective action to have knowledge regarding the victims and beneficiaries of external effects only) than in the case of mechanisms based on Pigovian taxes that equate the social marginal benefits and costs. Neither mechanism is, however, incentive compatible. But as was mentioned at the end of Section 4.1, the feasible sets of many incentive compatible mechanisms in the literature take the form of government proposals. Thus, for quasi-linear preferences, the collective action could implement, at every price vector that it faces, a Clarke-Groves mechanism to generate its demand for the public commodity. The resulting CCE will be incentive compatible. For more general economies, we conjecture that Foley's mechanism could also provide a framework of analysis for studying second-best optimal incentive compatible non-linear taxes/transfers for financing public commodities. Combined with redistributional considerations, this puts us in the general class of second-best policies based on multi-dimensional uncertainties: here, the free rider problem and incentives of agents to incorrectly reveal their true abilities.[36]

Appendix

Let $Y \in \mathbf{R}^n$ and $y \in Cl\ Y$.

Lemma A.1: *$int\ T(Y, y) = \{x \in \mathbf{R}^n | \exists\ \epsilon > 0,\ \eta > 0,\ \delta > 0$ such that, $\forall\ \lambda \in [0, \eta],\ \{y'\} + \lambda Cl\ \mathcal{N}_\epsilon(x) \subseteq Cl\ Y$ for all $y' \in (Cl\ Y \cap\ Cl\ \mathcal{N}_\delta(y))\}$.*

Remark: $N(Y, y) = T(Y, y)^- = (int\ T(Y, y))^-$.[37]

Lemma A.2: *Let $Y = \{\hat{y} \in \mathbf{R}^n | f(\hat{y}) \leq 0\}$, where f is continuous. If f is differentiable at y and $f(y) = 0$, then $T(Y, y) = \{x \in \mathbf{R}^n | x \cdot \nabla f(y) \leq 0\}$, and $N(Y, y) = \{p \in \mathbf{R}^n | p = \lambda \nabla f(y), \lambda \geq 0\}$.*

Lemma A.3: *Let $Y^i \in \mathbf{R}^n$, $int\ Y^i \neq \emptyset$, $i = 1, \ldots, m$, and $y \in \cap_i Cl\ Y^i$. Then $\cap_i int\ T(Y^i, y^i) \subseteq int\ T(\cap_i Y^i, y)$.*

Lemma A.4: *Let $K_0, \ldots, K_{p-1}$, be p open and non-empty convex cones and K_p be a convex cone with vertex 0^n in $\mathbf{R}^n$. Then $\bigcap_{i=0}^p K_i = \emptyset$ iff for all $i = 0, \ldots, p$, $\exists\ q^i \in K_i^-$ with $q^i \neq 0^n$ for some $i = 0, \ldots, p$ such that $\sum_i q^i = 0^n$.*[38]

Lemma A.5: *Let $Y := \prod_{i=1}^l Y^i$, and $y := \langle y^1, \ldots, y^l \rangle \in Y$, where $y^i \in Cl\ Y^i \subseteq \mathbf{R}^n, i = 1, \ldots, l$. Then $int\ T(Y, y) = \prod_{i=1}^l int\ T(Y^i, y^i)$, and $N(Y, y) = \prod_{i=1}^l N(Y^i, y^i)$.*

Lemma A.6: *Suppose Y is convex and $int\ Y \neq \emptyset$. Then $int\ T(Y, y) \neq \emptyset$.*

Lemma A.7: *Suppose Y is convex and y belongs to the boundary of Y. Then for all $a \in N(Y, y)$, the hyperplane with normal a and constant $a \cdot y$ is a supporting hyperplane to Y at y.*

Proof of $(1')$ to $(7')$ in Lemma 1:

Recall that a typical $s \in \mathcal{A}$ has the structure $s = \langle (x^h)_h, (y^i)_i, (z^j)_j \rangle$. Employing Lemmas A.2 and A.5 we describe the structures of the following vectors in $\mathcal{A}$:

(i) Employing Lemma A.5, for all h, the component x^h of $-\overset{*}{\sigma}{}_x^h$ (let us denote it by $-\overset{*}{\rho}{}_x^h$) belongs to the $N(R^h(\overset{*}{u}{}^h), \overset{*}{x}{}^h)$ and all other components of $-\overset{*}{\sigma}{}_x^h$ are zero. Similarly, for all

[36] For an example of second best policies based on two dimensional uncertainties, see Beaudry, Blackorby, and Szalay [2006].

[37] We denote the negative polar cone of a set A by A^-.

[38] See Guesnerie [1995], p. 281.

i, the component y^i of $\overset{*}{\sigma}{}^i_y$ (let us denote it by $\overset{*}{\rho}{}^i_y$) belongs to the $N(Y^i, \overset{*}{y}{}^i)$ and all other components of $\overset{*}{\sigma}{}^i_y$ are zero and, for all j, the component y^j of $\overset{*}{\sigma}{}^j_z$ (let us denote it by $\overset{*}{\rho}{}^j_z$) belongs to the $N(Z^j, \overset{*}{z}{}^j)$ and all other components of $\overset{*}{\sigma}{}^j_z$ are zero.

(ii) Employing Lemma A.2, for all α, ψ^α_o is $\lambda_\alpha \mathbf{1}_\alpha$ where λ_α is a scalar and $\mathbf{1}_\alpha \in \mathcal{A}$ is a vector with $x^h_o(\alpha)th$ element equal to 1 for all h, $y^i_o(\alpha)th$ and $z^j_o(\alpha)th$ elements being -1 for all i and j, and all other elements equal to zero.

(iii) For all h, γ, and j, $\psi^h_{x_g(\gamma)}$ is $\lambda^h_{x_g(\gamma)}\mathbf{1}^h_{x_g(\gamma)}$ where $\lambda^h_{x_g(\gamma)}$ is a scalar and $\mathbf{1}^h_{x_g(\gamma)} \in \mathcal{A}$ is a vector with $x_g(\gamma)th$ and $z^j_g(\gamma)th$ elements being 1 and -1, respectively, while all other elements are zero. Similarly, we can describe the structure of the vectors $\psi^h_{x_n(\beta)}$, $\psi^i_{y_g(\gamma)}$, and $\psi^i_{y_n(\beta)}$ for all i, h, γ, and β.

Consider

$$\sum_h \overset{*}{\sigma}{}^h_x + \sum_i \overset{*}{\sigma}{}^i_y + \sum_j \overset{*}{\sigma}{}^j_z + \sum_\alpha \psi^\alpha_o + \sum_{h,\gamma} \psi^h_{x_g(\gamma)} + \sum_{h,\beta} \psi^h_{x_n(\beta)} + \sum_{i,\gamma} \psi^i_{y_g(\gamma)} + \sum_{i,\beta} \psi^i_{y_n(\beta)} = 0. \quad (5.1)$$

Working through element-by-element of the left-hand side of (5.1), we find that

$$\begin{aligned} &-\overset{*}{\rho}{}^h_{x_o}(\alpha) = \overset{*}{\rho}{}^i_{y_o}(\alpha) = \overset{*}{\rho}{}^j_{z_o}(\alpha) = \lambda_\alpha \ \forall \alpha;\ \overset{*}{\rho}{}^h_{x_g(\gamma)} = \lambda^h_{x_g(\gamma)} \ \forall h, \gamma;\ \overset{*}{\rho}{}^h_{x_n(\beta)} = \lambda^h_{x_n(\beta)} \ \forall h, \beta; \\ &\overset{*}{\rho}{}^i_{y_g(\gamma)} = -\lambda^i_{y_g(\gamma)} \ \forall i, \gamma;\ \overset{*}{\rho}{}^i_{y_n(\beta)} = -\lambda^i_{y_n(\beta)} \ \forall n, \beta; \text{ and} \end{aligned} \quad (5.2)$$

$$\overset{*}{\rho}{}^j_{z_g(\gamma)} - \sum_i \lambda^i_{y_g(\gamma)} - \sum_h \lambda^h_{x_g(\gamma)} = 0 \ \forall \gamma, j \text{ and } \overset{*}{\rho}{}^j_{y_n(\beta)} - \sum_i \lambda^i_{y_n(\beta)} - \sum_h \lambda^h_{x_n(\beta)} = 0 \ \forall \beta, j. \quad (5.3)$$

Conclusions of Lemma 2 follow from substituting from (5.2) into (5.3).

References

Arrow, K. J. [1969], "Issues Pertinent to the Choice of Market Versus Non-Market Allocations," *The Analysis and Evaluation of Public Expenditures: The P. P. B. System,* Joint Committee of the Congress of the United States, Washington, D.C. : 47–64.

Baumol, W. J. [1972], "On Taxation and the Control of Externalities", *The American Economic Review*, 62: 307–322.

Baumol, W. J., and Bradford D. F. [1972], "Detrimental Externalities and nonconvexities of the Production Set", *Economica*, 39: 160–176.

Beato P., and A. Mas-Colell [1985], "On Marginal Cost Pricing Equilibria with Given Tax-Subsidy Rules," *Journal of Economic Theory* 37: 356–365.

Beaudry P., C. Blackorby, and D. Szalay [2006], "Taxes and Employment Subsidies in Optimal Redistribution Programs," A Draft.

Bonniseau J. M., and B. Cornet [1990] "Existence of Marginal Cost pricing Equilibria: The Nonsmooth Case," *International Economic Review*, 31: 685–708.

Boyd III, J. H., and J. P. Conley [1997], "Fundamental Nonconvexities in Arrovian Markets and the Coasian Solution to the Problem of Externalities," *Journal of Economic Theory* 72: 388–407.

Brown, D., and G. Heal [1978], "Equity, Efficiency, and Increasing Returns," *Review of economic Studies* 46: 571–585.

Clarke, F. H. [1975], "Generalized Gradients and Applications," *Transactions of the American Mathematical Society* 205: 247–262.

Clarke, F. H. [1983], *Optimization and Nonsmooth Analysis*, New York: Wiley.

Clarke, F.H. [1989], *Methods of Dynamic and Nonsmooth Optimization*, Montpelier, Vermont: Capital City Press.

Coase, R. [1960], "The Problem of Social Cost," *Journal of Law and Economics,* 3: 1–44.

Conley J. P., and S. C. Smith [2002], "Coasian Equilibrium," A Draft.

Cornet, B. [1989], "Existence of Equilibria in Economies with Increasing Returns", *Contributions to Operations Research and Economics: The Twentieth Anniversary of CORE*, edited by: Cornet B. and Tulkens H., Cambridge, Massachusetts: MIT Press.

Debreu, G. [1959], *Theory of Value*, Cowles Foundation Monograph 17: Yale University.

Diewert, W. E. [1974], "Applications of Duality Theory," *Frontiers of Quantitative Economics*, Volume II, edited by: Intrilligator M. D. and Kendrick D. A., North Holland Publishing Company.

Foley, D. K. [1967], *Resource Allocation and the Public Sector*, Yale Economic Essays, 7: 43–98, Yale University.

Foley, D. K. [1970], "Lindahl Solutions and the Core of an Economy with Public Goods," *Econometrica,* 38: 66–72.

Ghosal, S., and H. M. Polemarchakis [1999], "Exchange and Optimality," *Economic Theory,* 13: 629–42.

Guesnerie, R. [1975], "Pareto Optimality in Nonconvex Economies," *Econometrica* 43: 1–29.

Guesnerie, R. [1995], *A Contribution to the Pure Theory of Taxation*, Econometric Society Monographs, Cambridge, Cambridge University Press.

Hammond, P., and A. Villar [1998], "Efficiency with Non-Convexities: Extending the "Scandinavian Consensus" Approaches," *Scandinavian Journal of Economics*, 100: pp. 11-32.

Heal, G. [1999], *The Economics of Increasing Returns*, Edward Elger.

Hurwicz, L. [1999], "Revisiting Externalities," *Journal of Public Economic Theory*, 1: 225–245.

Khan, M. A., and R. Vohra [1987], "An Extension of the Second Welfare Theorem to Economies with Nonconvexities and Public Goods," *The Quarterly Journal of Economics* 102: 223–242.

Malinvaud, E. [1985], *Lectures in Microeconomic Theory*, Amsterdam, North Holland Publishing Company.

McFadden, D. [1978], "Cost, Revenue, and Profit Functions," *Production Economics: A Dual Approach to Theory and Applications*, Volume I, edited by: Fuss M. and McFadden D., Amsterdam, North Holland Publishing Company.

Milleron, J.C., [1972] "Theory of Value with Public Goods," *Journal of Economic Theory*, 5: 419–477.

Murty, S., [2006] "Externalities and Fundamental Nonconvexities: A Reconciliation of Approaches to General Equilibrium Externality Modeling and Implications for Decentralization," *The Warwick Economic Research Paper Series*, No. 756.

Otani, Y., and J. Sicilian [1977], "Externalities and Problems of Nonconvexity and Overhead Costs in Welfare Economics," *Journal of Economic Theory* 14: 239–251.

Pigou, A. C. [1932], *The Economics of Welfare*, London, Macmillan.

Quinzii, M. [1993], *Increasing Returns and Economic Efficiency*, Oxford University Press.

Rockafellar, R.T. [1978], "Clarke's Tangent Cones and the Boundaries of Closed Sets in $\mathbf{R}^n$," *Nonlinear Analysis, Theory and Applications* 3:145–154.

Starrett, D. [1972], "Fundamental Nonconvexities in the Theory of Externalities," *Journal of Economic Theory* 4: 180–199.

Wicksell, K. [1896], "A New Principle of Just Taxation," *Classics In The Theory of Public Finance*, edited by: Musgrave R. and Peacock A., [1962] Macmillan & Co. Ltd., London.