

# Existence of competitive equilibrium in an optimal growth model with elastic labor supply and smoothness of the policy function\*

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## Abstract

We prove the existence of competitive equilibrium and the smoothness of policy function in an optimal growth model with elastic labor supply by using a simple method. Our approach is based on the result of existence of Lagrange multipliers and their representation as a summable sequence due to Le Van and Saglam [2004] to define the sequence of prices and wages. The proof of existence of equilibrium we give is more simple than in Le Van and Vailakis [2004] and requires less stringent assumptions (neither Inada conditions for the utility function and the production function nor constant return to scale for the production function nor strict concavity). We also prove the differentiability of the policy function at a stationary optimal stock where the derivative of the policy function equals the smaller characteristic root in absolute value associated with Euler equation. Conditions for differentiability of the policy function have so far been assumed in the literature.

**Keywords:** Lagrange multipliers, Competitive equilibrium, Elastic labor supply, Optimal growth, Euler equation.

**JEL Classification:** *C61, D51, E13, O41*

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# 1 Introduction

The optimal growth model due to Cass and Koopmans is one of the main frameworks in macroeconomics. While variations of the model with inelastic labor supply are used widely in growth theory, the version with elastic labor supply is used in business cycle models, both for exogenous and endogenous fluctuations. Despite the central place of the model with elastic labor supply in dynamic general equilibrium models of macroeconomics, existence of competitive equilibrium in general settings has proved to be a challenge. Results of existence of equilibrium (Coleman (1997), Datta, *et al* (2002), Greenwood and Huffman (1995), Le Van and Vailakis (2004)) use strong conditions for existence. This paper establishes existence of equilibrium under very weak conditions: neither Inada conditions, nor strict concavity, nor constant returns to scale, nor restrictions on cross-partial of the utility functions. The paper also establishes differentiability of the policy function at the stationary stock. Unlike other papers which address this second issue (Mitra (2000), Santos (1991)) we do not need to assume additional conditions on the reduced form model, and show that the differentiability follows from the primitives of the model.

The approach taken in this paper is a direct method based on existence of Lagrange multipliers to the optimal problem and their representation as a summable sequence. This problem with inelastic labor supply was considered by Le Van and Saglam (2004). This approach uses a separation argument where the multipliers are represented in the dual space  $(l^\infty)'$  of the space of bounded sequences  $l^\infty$ . While one would like the multipliers and prices to lie in  $l^1$ , it is not the dual space (Debreu (1950)). Previous work following the work of Peleg and Yaari (1970), the representation theorems followed separation arguments applied to arbitrary vector spaces (see Bewley (1972), Majumdar (1972), Aliprantis et al. (1997), Dana and Le Van (1992)). The Le Van and Saglam (2004) approach also uses a separation argument but imposes restrictions on the asymptotic behavior of the objective functional and constraint functions which are easily shown to be satisfied in standard models. This is related to Dechert (1982).

There is a difficulty in going from the inelastic labor supply to the elastic labor supply model: while one can show that the optimal capital stock

is strictly positive, without assuming Inada conditions, one cannot be sure that the optimal labor supply sequence is strictly positive. Thus, the paper by Le Van and Vailakis (2004) which took the approach of decentralizing the optimal solution via prices as marginal utilities had to make additional strong conditions on the utility function to ensure that the labor supply sequence remains strictly positive. As we show, following Le Van and Saglam (2004), that the Lagrange multipliers to the social planners problem are a summable sequence, we can directly use these to decentralize the optimal solution and not have to make strong assumptions to ensure interiority of the optimal plan. Thus, the Inada conditions do not have to be assumed. As this approach does not require strict concavity, these strong assumptions on utility functions can be dropped. This is especially important as one important specification of preferences in applied macroeconomics models is that of linear utility of leisure where strict concavity is violated. This specification also results in the planners problem in models with indivisible labor (Hansen (1985), Rogerson (1988)). Furthermore, there is no need to make any assumption on cross-partial derivatives of the utility function as in Coleman (1997), Datta et al (2002), and Greenwood and Huffman (1995). Thus, whether labor supply is backward bending or not, and whether consumption is an inferior or not plays no role in existence of equilibrium. As a separation argument is used to generate the Lagrange multipliers, what is important is concavity and not also constant returns to scale of the production function.

The second main result of the paper is to show under standard assumptions on technology and preferences, the policy function is differentiable at the stationary optimal capital stock and the derivative is equal to the characteristic root with the smallest absolute value of the Ramsey-Euler equations. This result is important as one can use it to show stability properties of the model (Mitra and Nishimura (2005)). We show that under the standard assumptions, condition C1 in Mitra (2000) is satisfied, and thus the results there can be applied. This condition is a restriction on the second derivatives of the value function (Santos (1991) also has conditions on the second derivatives of the value function to show differentiability of the value function). Thus, for standard applied macroeconomic models with elastic labor supply one can check stability by studying derivatives of the policy function.

The organization of the paper is as follows. Section 2 provides the sufficient conditions on the objective function and the constraint functions so that Lagrangean multipliers can be presented by an  $l_+^1$  sequence of multipliers in optimal growth model with leisure in the utility function. In section 3, we prove the existence of competitive equilibrium in a model with a representative agent by using these multipliers as sequences of price and wage systems. Section 4 establishes differentiability of the policy function at the stationary stock. Section 5 concludes.

## 2 Lagrange multipliers in optimal growth model

Consider an economy in which a representative consumer has preferences defined over processes of consumption and leisure described by the utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t).$$

In each period, the consumer faces two resource constraints given by

$$\begin{aligned} c_t + k_{t+1} &\leq F(k_t, L_t) + (1 - \delta)k_t, \\ l_t + L_t &= 1, \forall t \end{aligned}$$

where  $F$  is the production function,  $\delta \in (0, 1)$  is the depreciation rate of capital stock and  $L_t$  is labor. These constraints restrict allocations of commodities and time for the leisure.

Formally, the problem of the representative consumer is stated as follows:

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\ \text{s.t. } &c_t + k_{t+1} \leq F(k_t, 1 - l_t) + (1 - \delta)k_t, \forall t \geq 0 \\ &c_t \geq 0, k_t \geq 0, l_t \geq 0, 1 - l_t \geq 0, \forall t \geq 0 \\ &k_0 \geq 0 \text{ is given.} \end{aligned}$$

We make a set of assumptions imposed on preferences and production technology. The assumptions on period utility function  $u : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  are as follows:

**Assumption U1:**  $u$  is continuous, concave, increasing on  $\mathbb{R}_+^2$  and strictly increasing on  $\mathbb{R}_{++}^2$ .

**Assumption U2:**  $u(0, 0) = 0$ .

The assumptions on the production function  $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  are as follows:

**Assumption F1:**  $F$  is continuous, concave, increasing on  $\mathbb{R}_+^2$  and strictly increasing on  $\mathbb{R}_{++}^2$ .

**Assumption F2:**  $F(0, 0) = 0, \lim_{k \rightarrow 0} F_k(k, 1) > \delta, \lim_{k \rightarrow +\infty} F_k(k, 1) < \delta$ .

The assumptions U1, U2, F1 are standard and we do not assume strict concavity of the utility and production functions. There are also no Inada conditions assumed. Assumption F2 is a weak assumption to ensure that there is a maximum sustainable capital stock, and thus the sequence of capital is bounded.

We say that a sequence  $(c_t, k_t, l_t)_{t=0,1,\dots,\infty}$  is feasible from  $k_0$  if it satisfies the constraints

$$\begin{aligned} \forall t &\geq 0, c_t + k_{t+1} \leq F(k_t, 1 - l_t) + (1 - \delta)k_t \\ c_t &\geq 0, k_t \geq 0, l_t \geq 0, 1 - l_t \geq 0, \\ k_0 &> 0 \text{ is given.} \end{aligned}$$

It is easy to check that, for any initial condition  $k_0 > 0$ , a sequence  $\mathbf{k} = (k_t)_{t=0}^\infty$  is feasible iff  $0 \leq k_{t+1} \leq F(k_t, 1) + (1 - \delta)k_t$  for all  $t$ . The class of feasible capital paths is denoted by  $\Pi(k_0)$ . A pair of consumption-leisure sequences  $(\mathbf{c}, \mathbf{l}) = (c_t, l_t)_{t=0}^\infty$  is feasible from  $k_0 > 0$  if there exists a sequence  $\mathbf{k} \in \Pi(k_0)$  that satisfies  $0 \leq c_t + k_{t+1} \leq F(k_t, 1 - l_t) + (1 - \delta)k_t$  and  $0 \leq l_t \leq 1$  for all  $t$ .

Define  $f(k_t, L_t) = F(k_t, L_t) + (1 - \delta)k_t$ . Assumption **F2** implies that

$$\begin{aligned} f_k(+\infty, 1) &= F_k(+\infty, 1) + (1 - \delta) < 1 \\ f_k(0, 1) &= F_k(0, 1) + (1 - \delta) > 1. \end{aligned}$$

From above, it follows that there exists  $\bar{k} > 0$  such that: (i)  $f(\bar{k}, 1) = \bar{k}$ , (ii)  $k > \bar{k}$  implies  $f(k, 1) < k$ , (iii)  $k < \bar{k}$  implies  $f(k, 1) > k$ . Therefore for any  $\mathbf{k} \in \Pi(k_0)$ , we have  $0 \leq k_t \leq \max(k_0, \bar{k})$ . Thus,  $\mathbf{k} \in l_+^\infty$  which in turn implies  $\mathbf{c} \in l_+^\infty$ , if  $(\mathbf{c}, \mathbf{k})$  is feasible from  $k_0$ . Denote  $\mathbf{x} = (\mathbf{c}, \mathbf{k}, \mathbf{l})$  and  $\mathcal{F}(\mathbf{x}) = -\sum_{t=0}^\infty \beta^t u(c_t, l_t)$ ,  $\Phi_t^1(\mathbf{x}) = c_t + k_{t+1} - f(k_t, 1 - l_t)$ ,  $\Phi_t^2(\mathbf{x}) = -c_t$ ,

$\Phi_t^3(\mathbf{x}) = -k_t$ ,  $\Phi_t^4(\mathbf{x}) = -l_t$ ,  $\Phi_t^5(\mathbf{x}) = l_t - 1$ ,  $\forall t$ ,  $\Phi_t = (\Phi_t^1, \Phi_t^2, \Phi_{t+1}^3, \Phi_t^4, \Phi_t^5)$ ,  
 $\forall t$ . The social planner's problem can be written as:

$$\min \mathcal{F}(\mathbf{x}) \text{ s.t. } \Phi(\mathbf{x}) \leq \mathbf{0}, \mathbf{x} \in l_+^\infty \times l_+^\infty \times l_+^\infty$$

$$\text{where } \mathcal{F} : l_+^\infty \times l_+^\infty \times l_+^\infty \rightarrow \mathbb{R} \cup \{+\infty\}$$

$$\Phi = (\Phi_t)_{t=0 \dots \infty} : l_+^\infty \times l_+^\infty \times l_+^\infty \rightarrow \mathbb{R} \cup \{+\infty\}$$

$$\text{Let } C = \text{dom}(\mathcal{F}) = \{\mathbf{x} \in l_+^\infty \times l_+^\infty \times l_+^\infty \mid \mathcal{F}(\mathbf{x}) < +\infty\}$$

$$\Gamma = \text{dom}(\Phi) = \{\mathbf{x} \in l_+^\infty \times l_+^\infty \times l_+^\infty \mid \Phi_t(\mathbf{x}) < +\infty, \forall t.\}$$

**Proposition 1** Let  $\mathbf{x}, \mathbf{y} \in l_+^\infty \times l_+^\infty \times l_+^\infty, T \in \mathbb{N}$ . Define

$$x_t^T(\mathbf{x}, \mathbf{y}) = \begin{cases} x_t & \text{if } t \leq T \\ y_t & \text{if } t > T \end{cases}.$$

Suppose that the two following assumptions are satisfied:

**T1:** If  $\mathbf{x} \in C$ ,  $\mathbf{y} \in l_+^\infty \times l_+^\infty \times l_+^\infty$  satisfy  $\forall T \geq T_0$ ,  $\mathbf{x}^T(\mathbf{x}, \mathbf{y}) \in C$ , then  $\mathcal{F}(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) \rightarrow \mathcal{F}(\mathbf{x})$  when  $T \rightarrow \infty$ .

**T2:** If  $\mathbf{x} \in \Gamma$ ,  $\mathbf{y} \in \Gamma$  and  $\mathbf{x}^T(\mathbf{x}, \mathbf{y}) \in \Gamma, \forall T \geq T_0$ , then

a)  $\Phi_t(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) \rightarrow \Phi_t(\mathbf{x})$  as  $T \rightarrow \infty$

b)  $\exists M$  s.t.  $\forall T \geq T_0$ ,  $\|\Phi_t(\mathbf{x}^T(\mathbf{x}, \mathbf{y}))\| \leq M$

c)  $\forall N \geq T_0, \lim_{t \rightarrow \infty} [\Phi_t(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) - \Phi_t(\mathbf{y})] = 0$ .

Let  $\mathbf{x}^*$  be a solution to (P) and  $\mathbf{x}^0 \in C$  satisfies the Slater condition:

$$\sup_t \Phi_t(\mathbf{x}^0) < 0.$$

Suppose  $\mathbf{x}^T(\mathbf{x}^*, \mathbf{x}^0) \in C \cap \Gamma$ . Then, there exists  $\Lambda \in l_+^1 \setminus \{0\}$  such that

$$\mathcal{F}(\mathbf{x}) + \Lambda \Phi(\mathbf{x}) \geq \mathcal{F}(\mathbf{x}^*) + \Lambda \Phi(\mathbf{x}^*), \forall \mathbf{x} \in (C \cap \Gamma)$$

and  $\Lambda \Phi(\mathbf{x}^*) = 0$ .

**Proof:** It is easy to see that  $l_+^\infty \times l_+^\infty \times l_+^\infty$  is isomorphic with  $l_+^\infty$ , since, for example, there exists an isomorphism

$$\Pi : l_+^\infty \rightarrow l_+^\infty \times l_+^\infty \times l_+^\infty,$$

$$\Pi(\mathbf{x}) = ((x_0, x_3, x_6, \dots)(x_1, x_4, x_7, \dots), (x_2, x_5, x_8, \dots))$$

and

$$\Pi^{-1}(\mathbf{u}, \mathbf{v}, \mathbf{s}) = (u_0, v_0, s_0, u_1, v_1, s_1, u_2, v_2, s_2, \dots)$$

Thus, there exists an isomorphism  $\Pi' : (l_+^\infty \times l_+^\infty \times l_+^\infty)' \rightarrow (l_+^\infty)'$ . It follows from Theorem1 in Le Van - Saglam [2004] that there exists  $\bar{\mathbf{\Lambda}} \in (l_+^\infty \times l_+^\infty \times l_+^\infty)'$ . Let  $\Lambda = \Pi'(\bar{\mathbf{\Lambda}}) \in (l_+^\infty)'$ . Then, the results are derived by the analogous arguments where a standard separation theorem used<sup>1</sup> as in the Theorem2 in Le Van - Saglam [2004]. ■

**Proposition 2** *If  $\mathbf{x}^* = (\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*)$  is a solution to the following problem:*

$$\min - \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \quad (P)$$

$$\begin{aligned} s.t. \quad & c_t + k_{t+1} - f(k_t, 1 - l_t) \leq 0, \\ & -c_t \leq 0, \quad -k_t \leq 0, \quad 0 \leq l_t \leq 1, \end{aligned}$$

*then there exists  $\lambda = (\lambda^1, \lambda^2, \lambda^3, \lambda^4, \lambda^5) \in l_+^1 \times l_+^1 \times l_+^1 \times l_+^1 \times l_+^1, \lambda \neq \mathbf{0}$  such that:  $\forall \mathbf{x} = (\mathbf{c}, \mathbf{k}, \mathbf{l}) \in l_+^\infty \times l_+^\infty \times l_+^\infty$*

$$\begin{aligned} & \sum_{t=0}^{\infty} \beta^t u(c_t^*, l_t^*) - \sum_{t=0}^{\infty} \lambda_t^1 (c_t^* + k_{t+1}^* - f(k_t^*, 1 - l_t^*)) \\ & + \sum_{t=0}^{\infty} \lambda_t^2 c_t^* + \sum_{t=0}^{\infty} \lambda_t^3 k_t^* + \sum_{t=0}^{\infty} \lambda_t^4 l_t^* + \sum_{t=0}^{\infty} \lambda_t^5 (1 - l_t^*) \\ \geq & \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) - \sum_{t=0}^{\infty} \lambda_t^1 (c_t + k_{t+1} - f(k_t, 1 - l_t)) \\ & + \sum_{t=0}^{\infty} \lambda_t^2 c_t + \sum_{t=0}^{\infty} \lambda_t^3 k_t + \sum_{t=0}^{\infty} \lambda_t^4 l_t + \sum_{t=0}^{\infty} \lambda_t^5 (1 - l_t) \quad (1) \end{aligned}$$

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<sup>1</sup>As the Remark 6.1.1 in LeVan and Dana [2003], assumption  $f_k(0, 1) > 1$  is equivalent to the Adequacy Assumption in Bewley (1972) and this assumption is crucial to have equilibrium prices in  $l_+^1$  since it implies that the production set has an interior point. Subsequently, it allows using a separation theorem in the infinite dimensional space to create Lagrange multipliers.

$$\lambda_t^1(c_t^* + k_{t+1}^* - f(k_t^*, 1 - l_t^*)) = 0 \quad (2)$$

$$\lambda_t^2 c_t^* = 0 \quad (3)$$

$$\lambda_t^3 k_t^* = 0 \quad (4)$$

$$\lambda_t^4 l_t^* = 0 \quad (5)$$

$$\lambda_t^5(1 - l_t^*) = 0 \quad (6)$$

$$0 \in \beta^t \partial_1 u(c_t^*, l_t^*) - \{\lambda_t^1\} + \{\lambda_t^2\} \quad (7)$$

$$0 \in \beta^t \partial_2 u(c_t^*, l_t^*) - \lambda_t^1 \partial_2 f(k_t^*, L_t^*) + \{\lambda_t^4\} - \{\lambda_t^5\} \quad (8)$$

$$0 \in \lambda_t^1 \partial_1 f(k_t^*, L_t^*) + \{\lambda_t^3\} - \{\lambda_{t-1}^1\} \quad (9)$$

where  $\partial_i u(c_t^*, l_t^*), \partial_i f(k_t^*, L_t^*)$  respectively denote the projection on the  $i^{\text{th}}$  component of the subdifferential of function  $u$  at  $(c_t^*, l_t^*)$  and the function  $f$  at  $(k_t^*, L_t^*)$ .

Note that **T1** holds when  $F$  is continuous in the product topology. **T2c** is satisfied if the asymptotically insensitivity, i.e., if  $x$  is changed only on a finitely many values the constraint value for large  $t$  does not change that much (Dechert (1982)). **T2c** is the asymptotically non-anticipatory assumption and requires  $\Phi$  to be nearly weak-\* continuous (Dechert (1982)). **T2b** holds when when  $\Gamma = \text{dom}(\Phi) = l^\infty$  and  $\Phi$  is continuous (see Dechert (1982), Le Van and Saglam (2004)).

**Proof:** We first check that the Slater condition holds. Indeed, since  $f'_k(0, 1) > 1$ , then for all  $k_0 > 0$ , there exists some  $0 < \widehat{k} < k_0$  such that:  $0 < \widehat{k} < f(\widehat{k}, 1)$  and  $0 < \widehat{k} < f(k_0, 1)$ . Thus, there exists two small positive numbers  $\varepsilon, \varepsilon_1$  such that:

$$0 < \widehat{k} + \varepsilon < f(\widehat{k}, 1 - \varepsilon_1) \text{ and } 0 < \widehat{k} + \varepsilon < f(k_0, 1 - \varepsilon_1).$$

Denote  $\mathbf{x}^0 = (\mathbf{c}^0, \mathbf{k}^0, \mathbf{l}^0)$  such that  $\mathbf{c}^0 = (\varepsilon, \varepsilon, \dots)$ ,  $\mathbf{k}^0 = (k_0, \widehat{k}, \widehat{k}, \dots)$ ,  $\mathbf{l}^0 = (\varepsilon_1, \varepsilon_1, \dots)$ . We have

$$\begin{aligned} \Phi_0^1(\mathbf{x}^0) &= c_0 + k_1 - f(k_0, 1 - l_0) \\ &= \varepsilon + \widehat{k} - f(k_0, 1 - \varepsilon_1) < 0 \\ \Phi_1^1(\mathbf{x}^0) &= c_1 + k_2 - f(k_1, 1 - l_1) \\ &= \varepsilon + \widehat{k} - f(\widehat{k}, 1 - \varepsilon_1) < 0 \\ \Phi_t^1(\mathbf{x}^0) &= \varepsilon + \widehat{k} - f(\widehat{k}, 1 - \varepsilon_1) < 0, \quad \forall t \geq 2 \end{aligned}$$

$$\begin{aligned}\Phi_t^2(\mathbf{x}^0) &= -\varepsilon < 0, \quad \forall t \geq 0, \quad \Phi_0^3(\mathbf{x}^0) = -k_0 < 0. \\ \Phi_t^3(\mathbf{x}^0) &= -\widehat{k} < 0 \quad \forall t \geq 1, \quad \Phi_t^4(\mathbf{x}^0) = -\varepsilon_1 < 0, \quad \forall t \geq 0. \\ \Phi_t^5(\mathbf{x}^0) &= \varepsilon_1 - 1 < 0, \quad \forall t \geq 0\end{aligned}$$

Therefore the Slater condition is satisfied. Now, it is obvious that,  $\forall T$ ,  $\mathbf{x}^T(\mathbf{x}^*, \mathbf{x}^0)$  belongs to  $l_+^\infty \times l_+^\infty \times l_+^\infty$ . As in Le Van-Saglam 2004, Assumption **T2** is satisfied. We now check Assumption **T1**. For any  $\tilde{\mathbf{x}} \in C$ ,  $\tilde{\mathbf{x}} \in l_+^\infty \times l_+^\infty \times l_+^\infty$  such that for any  $T$ ,  $\mathbf{x}^T(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}) \in C$  we have

$$\mathcal{F}(\mathbf{x}^T(\tilde{\mathbf{x}}, \tilde{\mathbf{x}})) = -\sum_{t=0}^T \beta^t u(\tilde{c}_t, \tilde{l}_t) - \sum_{t=T+1}^{\infty} \beta^t u(\tilde{c}_t, \tilde{l}_t).$$

As  $\tilde{\mathbf{x}} \in l_+^\infty \times l_+^\infty \times l_+^\infty$ ,  $\sup_t |\tilde{c}_t| < +\infty$ , there exists  $m > 0, \forall t, |\tilde{c}_t| \leq m$ . Since  $\beta \in (0, 1)$  we have

$$\sum_{t=T+1}^{\infty} \beta^t u(m, 1) = u(m, 1) \sum_{t=T+1}^{\infty} \beta^t \rightarrow 0 \text{ as } T \rightarrow \infty.$$

Hence,  $\mathcal{F}(\mathbf{x}^T(\tilde{\mathbf{x}}, \tilde{\mathbf{x}})) \rightarrow \mathcal{F}(\tilde{\mathbf{x}})$  when  $T \rightarrow \infty$ . Taking account of the Proposition 1, we get (1) - (6).

Finally, we obtain (7) - (9) from the Kuhn-Tucker first-order conditions.

■

### 3 Competitive equilibrium

**Definition 1** *A competitive equilibrium for this model consists of an allocation  $\{\mathbf{c}^*, \mathbf{l}^*, \mathbf{k}^*, \mathbf{L}^*\} \in l_+^\infty \times l_+^\infty \times l_+^\infty \times l_+^\infty$ , a price sequence  $\mathbf{p}^* \in l_+^1$  for the consumption good, a wage sequence  $\mathbf{w}^* \in l_+^1$  for labor and a price  $r > 0$  for the initial capital stock  $k_0$  such that:*

*i)  $(\mathbf{c}^*, \mathbf{l}^*)$  is a solution to the problem*

$$\begin{aligned}\max \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\ \text{s.t.} \quad & \mathbf{p}^* \mathbf{c} \leq \mathbf{w}^* \mathbf{L} + \pi^* + rk_0\end{aligned}$$

where  $\pi^*$  is the maximum profit of the firm.

ii)  $(\mathbf{k}^*, \mathbf{L}^*)$  is a solution to the firm's problem

$$\begin{aligned} \pi^* &= \max \sum_{t=0}^{\infty} p_t^* [f(k_t, L_t) - k_{t+1}] - \sum_{t=0}^{\infty} w_t^* L_t - r k_0 \\ \text{s.t. } &0 \leq k_{t+1} \leq f(k_t, L_t), L_t \geq 0, \forall t. \end{aligned}$$

iii) Markets clear

$$\begin{aligned} \forall t, c_t^* + k_{t+1}^* &= f(k_t^*, L_t^*) \\ l_t^* + L_t^* &= 1 \text{ and } k_0^* = k_0 \end{aligned}$$

**Theorem 1** Let  $(\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*)$  solve Problem (P). Take

$$p_t^* = \lambda_t^1 \text{ for any } t \text{ and } r > 0.$$

There exists  $f_L(k_t^*, L_t^*) \in \partial_2 f(k_t^*, L_t^*)$  such that  $\{\mathbf{c}^*, \mathbf{k}^*, \mathbf{L}^*, \mathbf{p}^*, \mathbf{w}^*, r\}$  is a competitive equilibrium with  $w_t^* = \lambda_t^1 f_L(k_t^*, L_t^*)$ .

**Proof:** Consider  $\lambda = (\lambda^1, \lambda^2, \lambda^3, \lambda^4, \lambda^5)$  of Proposition 2. Conditions (7),(8),(9) in Proposition 2 show that  $\partial u(c_t^*, l_t^*)$  and  $\partial f(k_t^*, L_t^*)$  are nonempty and there exists  $u_c(c_t^*, l_t^*) \in \partial_1 u(c_t^*, l_t^*)$ ,  $u_l(c_t^*, l_t^*) \in \partial_2 u(c_t^*, l_t^*)$ ,  $f_k(k_t^*, L_t^*) \in \partial_1 f(k_t^*, L_t^*)$  and  $f_L(k_t^*, L_t^*) \in \partial_2 f(k_t^*, L_t^*)$  such that  $\forall t$

$$\beta^t u_c(c_t^*, l_t^*) - \lambda_t^1 + \lambda_t^2 = 0 \quad (10)$$

$$\beta^t u_l(c_t^*, l_t^*) - \lambda_t^1 f_L(k_t^*, L_t^*) + \lambda_t^4 - \lambda_t^5 = 0 \quad (11)$$

$$\lambda_t^1 f_k(k_t^*, L_t^*) + \lambda_t^3 - \lambda_{t-1}^1 = 0 \quad (12)$$

Define  $w_t^* = \lambda_t^1 f_L(k_t^*, L_t^*) < +\infty$ . We now prove that  $\mathbf{w}^* \in l_+^1$ .

We have

$$+\infty > \sum_{t=0}^{\infty} \beta^t u(c_t^*, l_t^*) - \sum_{t=0}^{\infty} \beta^t u(0, 0) \geq \sum_{t=0}^{\infty} \beta^t u_c(c_t^*, l_t^*) c_t^* + \sum_{t=0}^{\infty} \beta^t u_l(c_t^*, l_t^*) l_t^*,$$

which implies

$$\sum_{t=0}^{\infty} \beta^t u_l(c_t^*, l_t^*) l_t^* < +\infty, \quad (13)$$

and

$$+\infty > \sum_{t=0}^{\infty} \lambda_t^1 f(k_t^*, L_t^*) - \sum_{t=0}^{\infty} \lambda_t^1 f(0, 0) \geq \sum_{t=0}^{\infty} \lambda_t^1 f_k(k_t^*, L_t^*) k_t^* + \sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^*$$

which implies

$$\sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^* < +\infty. \quad (14)$$

Given  $T$ , we multiply (11) by  $L_t^*$  and sum up from 0 to  $T$ . We then obtain. Observe that

$$\forall T, \sum_{t=0}^T \beta^t u_l(c_t^*, l_t^*) L_t^* = \sum_{t=0}^T \lambda_t^1 f_L(k_t^*, L_t^*) L_t^* + \sum_{t=0}^T \lambda_t^5 L_t^* - \sum_{t=0}^T \lambda_t^4 L_t^*. \quad (15)$$

$$0 \leq \sum_{t=0}^{\infty} \lambda_t^5 L_t^* \leq \sum_{t=0}^{\infty} \lambda_t^5 < +\infty. \quad (16)$$

$$0 \leq \sum_{t=0}^{\infty} \lambda_t^4 L_t^* \leq \sum_{t=0}^{\infty} \lambda_t^4 < +\infty \quad (17)$$

Thus, since  $L_t^* = 1 - l_t^*$ , from (15), we get

$$\begin{aligned} \sum_{t=0}^T \beta^t u_l(c_t^*, l_t^*) &= \sum_{t=0}^T \beta^t u_l(c_t^*, l_t^*) l_t^* + \sum_{t=0}^T \lambda_t^1 f_L(k_t^*, L_t^*) L_t^* \\ &\quad + \sum_{t=0}^T \lambda_t^5 L_t^* - \sum_{t=0}^T \lambda_t^4 L_t^* \end{aligned}$$

Using (13),(14),(16),(17) and letting  $T \rightarrow \infty$ , we obtain

$$\begin{aligned} 0 &\leq \sum_{t=0}^{\infty} \beta^t u_l(c_t^*, l_t^*) = \sum_{t=0}^{\infty} \beta^t u_l(c_t^*, l_t^*) l_t^* + \sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^* \\ &\quad + \sum_{t=0}^{\infty} \lambda_t^5 L_t^* - \sum_{t=0}^{\infty} \lambda_t^4 L_t^* < +\infty \end{aligned}$$

Consequently, from (11),  $\sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*) < +\infty$  i.e.  $\mathbf{w}^* \in l_+^1$ . So, we have  $\{\mathbf{c}^*, \mathbf{l}^*, \mathbf{k}^*, \mathbf{L}^*\} \in l_+^{\infty} \times l_+^{\infty} \times l_+^{\infty} \times l_+^{\infty}$ , with  $\mathbf{p}^* \in l_+^1$  and  $\mathbf{w}^* \in l_+^1$ .

We now show that  $(\mathbf{k}^*, \mathbf{L}^*)$  is solution to the firm's problem.

Since  $p_t^* = \lambda_t^1$ ,  $w_t^* = \lambda_t^1 f_L(k_t^*, L_t^*)$ , we have

$$\pi^* = \sum_{t=0}^{\infty} \lambda_t^1 [f(k_t^*, L_t^*) - k_{t+1}^*] - \sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^* - rk_0$$

Let :

$$\begin{aligned} \Delta_T &= \sum_{t=0}^T \lambda_t^1 [f(k_t^*, L_t^*) - k_{t+1}^*] - \sum_{t=0}^T \lambda_t^1 f_L(k_t^*, L_t^*) L_t^* - rk_0 \\ &\quad - \left( \sum_{t=0}^T \lambda_t^1 [f(k_t, L_t) - k_{t+1}] - \sum_{t=0}^T \lambda_t^1 f_L(k_t^*, L_t^*) L_t - rk_0 \right). \end{aligned}$$

By the concavity of  $f$ , we get

$$\begin{aligned} \Delta_T &\geq \sum_{t=1}^T \lambda_t^1 f_k(k_t^*, L_t^*) (k_t^* - k_t) - \sum_{t=0}^T \lambda_t^1 (k_{t+1}^* - k_{t+1}) \\ &= [\lambda_1^1 f_k(k_1^*, L_1^*) - \lambda_0^1] (k_1^* - k_1) + \dots \\ &\quad + [\lambda_T^1 f_k(k_T^*, L_T^*) - \lambda_{T-1}^1] (k_T^* - k_T) - \lambda_T^1 (k_{T+1}^* - k_{T+1}). \end{aligned}$$

By (4) and (12), we have:  $\forall t = 1, 2, \dots, T$

$$[\lambda_t^1 f_k(k_t^*, L_t^*) - \lambda_{t-1}^1] (k_t^* - k_t) = -\lambda_t^3 (k_t^* - k_t) = \lambda_t^3 k_t \geq 0.$$

Thus,

$$\Delta_T \geq -\lambda_T^1 (k_{T+1}^* - k_{T+1}) = -\lambda_T^1 k_{T+1}^* + \lambda_T^1 k_{T+1} \geq -\lambda_T^1 k_{T+1}^*.$$

Since  $\lambda^1 \in l_+^1$ ,  $\sup_T k_{T+1}^* < +\infty$ , we have

$$\lim_{T \rightarrow +\infty} \Delta_T \geq \lim_{T \rightarrow +\infty} -\lambda_T^1 k_{T+1}^* = 0.$$

We have proved that the sequences  $(\mathbf{k}^*, \mathbf{L}^*)$  maximize the profit of the firm.

We now show that  $c^*$  solves the consumer's problem.

$$\text{Let } \{\mathbf{c}, \mathbf{L}\} \text{ satisfy } \sum_{t=0}^{\infty} \lambda_t^1 c_t \leq \sum_{t=0}^{\infty} w_t^* L_t + \pi^* + rk_0 \quad (18)$$

By the concavity of  $u$ , we have:

$$\Delta = \sum_{t=0}^{\infty} \beta^t u(c_t^*, l_t^*) - \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

$$\geq \sum_{t=0}^{\infty} \beta^t u_c(c_t^*, l_t^*)(c_t^* - c_t) + \sum_{t=0}^{\infty} \beta^t u_l(c_t^*, l_t^*)(l_t^* - l_t).$$

Combining (3),(6),(10),(11) yields

$$\begin{aligned} \Delta &\geq \sum_{t=0}^{\infty} (\lambda_t^1 - \lambda_t^2)(c_t^* - c_t) + \sum_{t=0}^{\infty} (\lambda_t^1 f_L(k_t^*, 1 - l_t^*) + \lambda_t^5 - \lambda_t^4)(l_t^* - l_t) \\ &= \sum_{t=0}^{\infty} \lambda_t^1 (c_t^* - c_t) + \sum_{t=0}^{\infty} \lambda_t^2 c_t - \sum_{t=0}^{\infty} \lambda_t^2 c_t^* + \sum_{t=0}^{\infty} (w_t^* + \lambda_t^5)(l_t^* - l_t) \\ &\quad - \sum_{t=0}^{\infty} \lambda_t^4 l_t^* + \sum_{t=0}^{\infty} \lambda_t^4 l_t \\ &\geq \sum_{t=0}^{\infty} \lambda_t^1 (c_t^* - c_t) + \sum_{t=0}^{\infty} (w_t^* + \lambda_t^5)(l_t^* - l_t) = \\ &\quad \sum_{t=0}^{\infty} \lambda_t^1 (c_t^* - c_t) + \sum_{t=0}^{\infty} w_t^* (l_t^* - l_t) + \sum_{t=0}^{\infty} \lambda_t^5 (1 - l_t) \\ &\geq \sum_{t=0}^{\infty} \lambda_t^1 (c_t^* - c_t) + \sum_{t=0}^{\infty} w_t^* (L_t - L_t^*) \end{aligned}$$

Since

$$\pi^* = \sum_{t=0}^{\infty} \lambda_t^1 c_t^* - \sum_{t=0}^{\infty} w_t^* L_t^* - rk_0,$$

it follows from (18) that

$$\begin{aligned} \Delta &\geq \sum_{t=0}^{\infty} p_t^* c_t^* - \sum_{t=0}^{\infty} w_t^* L_t^* - rk_0 - \left[ \sum_{t=0}^{\infty} p_t^* c_t - \sum_{t=0}^{\infty} w_t^* L_t - rk_0 \right] \\ &\geq \pi^* - \pi^* = 0 \end{aligned}$$

Consequently,  $\Delta \geq 0$  that means  $c^*$  solves the consumer's problem.

Finally, the market clears at every period, since  $\forall t, c_t^* + k_{t+1}^* = f(k_t^*, L_t^*)$  and  $1 - l_t^* = L_t^*$ . ■

## 4 Differentiability of the policy function

To show differentiability of the policy function we need additional set of assumptions imposed on preferences and production technology. These are standard in the macroeconomics literature. Assumption **U3** imposes Inada conditions, and **U4** is weaker than assuming normality (which requires in addition that  $\left(\frac{u_{ll}}{u_l} \leq \frac{u_{cl}}{u_c}\right)$ ). This condition just requires the marginal rate of substitution  $\frac{u_l}{u_c}$  be non-decreasing in  $c$ . When  $u_{cl} \geq 0$  it is trivially satisfied. It establishes monotonicity of the optimal capital path (see Aiyagari (1992), and the discussion in Le Van and Saglam (2004)). Assumptions **F3**, **F4** are standard in the literature.

**Assumption U3:**  $u$  is twice continuously differentiable on  $\mathbb{R}_{++}^2$  with partial derivatives satisfying the Inada conditions:  $\lim_{c \rightarrow 0} u_c(c, l) = +\infty, \forall l > 0$  and  $\lim_{l \rightarrow 0} u_l(c, l) = +\infty, \forall c > 0$ .

**Assumption U4:** For all  $c > 0$  and  $l > 0$ , the cross partial derivative  $u_{cl}$  has a constant sign. In addition, we require the first and second partial derivatives to verify the following condition:

$$\frac{u_{cc}}{u_c} \leq \frac{u_{cl}}{u_l}.$$

Moreover,  $u_{cc} < 0$  and  $u_{ll} < 0$ , for all  $c > 0$  and  $l > 0$ .

**Assumption F3:**  $F$  is twice continuously differentiable on  $\mathbb{R}_{++}^2$  with partial derivatives satisfying the Inada conditions:  $\lim_{k \rightarrow 0} F_k(k, 1) = +\infty$ ,  $\lim_{k \rightarrow +\infty} F_k(k, 1) < \delta$  and  $\lim_{L \rightarrow 0} F_L(k, L) = +\infty, \forall k > 0$ .

**Assumption F4:** For all  $k > 0$  and  $L > 0$ ,  $F_{kL} \geq 0$  and  $F_{kk} < 0, F_{LL} < 0$ .

In order to save notation, we write  $f(k_t, L_t) = F(k_t, L_t) + (1 - \delta)k_t$ . Observe that under the previous assumptions,  $\lim_{k \rightarrow 0} f_k(k, 1) = +\infty$ ,  $\lim_{k \rightarrow +\infty} f_k(k, 1) < 1$  and  $\lim_{L \rightarrow 0} f_L(k, L) = +\infty$ .

Let us rewrite the social planning problem (P) of determining a Pareto

efficient consumption-leisure allocation and production sequence

$$\begin{aligned}
& \max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\
& \text{s.t. } c_t + k_{t+1} \leq f(k_t, L_t), \forall t \\
& \quad l_t + L_t \leq 1, \forall t \\
& \quad c_t \geq 0, l_t \geq 0, k_t \geq 0, L_t \geq 0, \forall t \\
& \quad k_0 \geq 0 \text{ is given.}
\end{aligned}$$

Let  $(k, y)$  verify  $0 \leq y \leq f(k, 1)$ . Define the function  $V : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ :

$$\begin{aligned}
V(k, y) &= \max u(c, l) \\
& \text{s.t. } c + y \leq f(k, 1 - l) \\
& \quad c \geq 0, 0 \leq l \leq 1.
\end{aligned}$$

Consider the following problem (Q):

$$\begin{aligned}
& \max \sum_{t=0}^{\infty} \beta^t V(k_t, k_{t+1}) \\
& \text{s.t. } 0 \leq k_{t+1} \leq f(k_t, 1), \forall t \\
& \quad k_0 \geq 0 \text{ is given.}
\end{aligned}$$

It is easy to see that problems (P) and (Q) are equivalent.

The principle of optimality is formally stated in the following proposition. It will help us characterize basic properties of optimal paths.

**Proposition 3** *The value function solves the Bellman equation, i.e.*

$$\forall k_0 \geq 0, W(k_0) = \max\{V(k_0, k_1) + \beta W(k_1) : 0 \leq k_1 \leq f(k_0, 1)\},$$

and for all  $k_0 \geq 0$ , a feasible path  $\mathbf{k}$  is optimal, if and only if,

$$W(k_t) = V(k_t, k_{t+1}) + \beta W(k_{t+1})$$

holds for all  $t$ .

**Proof:** See Stokey and Lucas (1989, Ch. 4). ■

**Lemma 1** *Assume U1-U3 and F1-F3. Let  $k_0 > 0$  and denote by  $(\mathbf{c}^*, \mathbf{l}^*, \mathbf{k}^*)$  the optimal solution to problem (P). Then:*

- i)  $k_t^* > 0, \forall t.$*
- ii)  $c_t^* > 0$  and  $l_t^* > 0, \forall t.$*

**Proof:** See Appendix ■

Now we can prove that the optimal solution  $(\mathbf{c}^*, \mathbf{l}^*, \mathbf{k}^*)$  converges to a unique optimal steady state  $(c^s, l^s, k^s)$  which is unique and nontrivial .

**Proposition 4** *There exists a unique nontrivial steady state  $(c^s, l^s, k^s)$  that satisfies  $f_k(k^s, 1 - l^s) = \frac{1}{\beta}$  and  $c^s = f(k^s, 1 - l^s) - k^s.$*

**Proof:** For the single agent case, see Le Van and Vailakis (2004) and for the many agents, see Le Van, Nguyen and Vailakis (2007). ■

Let us define the optimal policy function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+ :$

$$g(k_0) = \arg \max \{V(k_0, k_1) + \beta W(k_1) : 0 \leq k_1 \leq f(k_0, 1)\}.$$

**Proposition 5** *Assume U1-U3 and F1-F3. The Euler equations hold for  $t \geq T$ , i.e.,*

$$u_c(c_t^*, l_t^*) = \beta u_c(c_{t+1}^*, l_{t+1}^*) f_k(k_{t+1}^*, L_{t+1}^*).$$

**Proof:** See Appendix ■

It follow from Euler equation that  $V_2(k_t^*, k_{t+1}^*) + \beta V_1(k_{t+1}^*, k_{t+2}^*) = 0.$  The characteristic equation associated with the Ramsey-Euler equation is

$$\beta \lambda^2 V_{21} + [V_{22} + \beta V_{11}] \lambda + V_{12} = 0.$$

The following Lemma play an important role to prove the differentiability of policy function at the stationary optimal stock. It is an assumption in Mitra (2000). Note that it is a restriction of derivatives of the value function and thus, as such it is not clear what underlying assumptions on the technology and preferences generate it. As we show in addition to standard differentiability, Inada, and concavity assumptions all that is needed is the non-decreasing marginal rate of substitution which ensures monotonicity of the optimal capital stock.

**Lemma 2** *Let  $k^s$  be a steady state. Then*

$$(1 + \beta)|V_{12}(k^s, k^s)| + \beta V_{11}(k^s, k^s) + V_{22}(k^s, k^s) < 0.$$

**Proof:** Let the function  $V$  be defined as before, i.e. given any  $(k, y)$ , such that,  $0 \leq y \leq f(k, 1)$ ,

$$\begin{aligned} V(k, y) &= \max u(c, l) \\ &\text{s.t. } c + y \leq f(k, 1 - l) \\ &\quad c \geq 0, 0 \leq l \leq 1. \end{aligned}$$

Note that the Slater condition is trivially satisfied for  $c = 0, l = 0$ , so there exist multipliers  $\lambda, \mu \in \mathbb{R}$  associated with the constraints  $c + y \leq f(k, 1 - l)$  and  $l \leq 1$ , such that,  $(c^*, l^*, \lambda, \mu)$  maximizes the associated Lagrangian. The Kuhn-Tucker first-order conditions are<sup>2</sup>:

$$\begin{aligned} 0 &\in \partial_1(u(c^*, l^*)) - \{\lambda\} \\ 0 &\in \partial_2(u(c^*, l^*)) - \lambda \partial_2(f(k, 1 - l^*)) - \{\mu\} \\ \lambda &\geq 0, \lambda [c^* + y - f(k, 1 - l^*)] = 0 \\ \mu &\geq 0, \mu (l^* - 1) = 0. \end{aligned}$$

The strict increasingness of  $u$  together with assumption **U2** imply that  $(c^*, l^*) \neq 0$ . Since  $u$  satisfies the Inada conditions, it follows that  $c^* > 0$  and  $l^* > 0$  (otherwise  $\partial_1(u(c^*, l^*))$  and  $\partial_2(u(c^*, l^*))$  will be empty). Condition 1 implies that  $\lambda > 0$ . Observe that the Inada condition on labor's marginal productivity implies that  $l^* < 1, \mu = 0$  (otherwise  $\partial_2(f(k, 1 - l^*))$  will be empty). We have

$$\begin{aligned} u_c(c^*, l^*) - \lambda &= 0 \\ u_l(c^*, l^*) - \lambda f_L(k, 1 - l^*) &= 0 \\ c^* + y - f(k, 1 - l^*) &= 0. \end{aligned}$$

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<sup>2</sup>For a concave function  $f$  defined on  $\mathbb{R}^n$ ,  $\partial f(x)$  denotes the subdifferential of  $f$  at  $x \in \mathbb{R}^n$ . Note also that  $\partial_i f(x)$  is the projection of  $\partial f(x)$  on the  $i_{th}$  component.

Uniqueness of the solution follows from the strict concavity of  $u$ . Differentiating the above equations gives:

$$\begin{aligned} u_{cc}dc^* + u_{cl}dl^* - d\lambda &= 0 \\ u_{cl}dc^* + u_{ll}dl^* - \lambda[f_{kL}dk - f_{LL}dl^*] - f_Ld\lambda &= 0 \\ dc^* - f_kdk + dy + f_Ldl^* &= 0 \end{aligned}$$

Since the optimal solution  $(\mathbf{c}^*, \mathbf{l}^*, \mathbf{k}^*)$  converges to a unique nontrivial optimal steady state  $(c^s, l^s, k^s)$ , by taking the limit, the above equations become

$$adc^s + bdl^s - d\lambda = 0 \quad (1)$$

$$bdc^s + [c + p]dl^s - f_Ld\lambda = \lambda f_{Lk}dk. \quad (2)$$

$$dc^s + f_Ldl^s = f_kdk - dy. \quad (3)$$

where

$$p = \lambda f_{LL}, a = u_{cc}, b = u_{cl}, c = u_{ll}.$$

Denote

$$V_{12}(k^s, k^s) = \widehat{V}_{12}, V_{11}(k^s, k^s) = \widehat{V}_{11}, V_{22}(k^s, k^s) = \widehat{V}_{22}.$$

We already have

$$\beta f_k(k^s, 1 - l^s) = 1,$$

$$\begin{aligned} \widehat{V}_1 &= \frac{\partial V(k, y)}{\partial k} = \lambda f_k(k^s, 1 - l^s) \\ \widehat{V}_2 &= \frac{\partial V(k, y)}{\partial y} = -\lambda. \end{aligned}$$

Thus,

$$\begin{aligned} \widehat{V}_{21} &= \frac{\partial^2 \widehat{V}}{\partial y \partial k} = -\frac{\partial \lambda}{\partial k}, \widehat{V}_{22} = \frac{\partial^2 \widehat{V}}{\partial y \partial y} = -\frac{\partial \lambda}{\partial y} \\ \widehat{V}_{11} &= \frac{\partial^2 \widehat{V}}{\partial k \partial k} = \frac{\partial(\lambda f_k(k, 1 - l^s))}{\partial k} = \\ &= \frac{\partial \lambda}{\partial k} f_k + \lambda(f_{kk} - f_{kL} \frac{\partial l^s}{\partial k}). \end{aligned}$$

It follows from (1),(2),(3) that

$$\begin{aligned} (f_L - \frac{b}{a})dl^s + \frac{1}{a}d\lambda &= f_k - dy \\ (c + p - \frac{b^2}{a})dl^s - (f_L - \frac{b}{a})d\lambda &= \lambda f_{Lk}dk \end{aligned}$$

Denote

$$\begin{aligned} x &= f_L - \frac{b}{a} = \frac{u_l}{u_c} - \frac{u_{cl}}{u_{cc}} \\ &= \frac{u_l}{u_{cc}} \left( \frac{u_{cc}}{u_c} - \frac{u_{cl}}{u_l} \right) \geq 0, \\ y &= c - \frac{b^2}{a} + p = \frac{ac - b^2}{a} + p < 0 \end{aligned}$$

Denote  $A = \det \begin{bmatrix} x & 1/a \\ y & -x \end{bmatrix} = -x^2 - \frac{y}{a} < 0$ , we have

$$\begin{aligned} dl^s &= \frac{-xf_k - \frac{\lambda f_{kL}}{a}}{A}dk + \frac{x}{A}dy, \\ d\lambda &= \frac{x\lambda f_{kL} - yf_k}{A}dk + \frac{y}{A}dy. \end{aligned}$$

Thus,

$$\frac{\partial l^s}{\partial k} = \frac{-xf_k - \frac{\lambda f_{kL}}{a}}{A} \geq 0, \quad (4)$$

$$\frac{\partial \lambda}{\partial k} = \frac{x\lambda f_{kL} - yf_k}{A}, \quad \frac{\partial \lambda}{\partial y} = \frac{y}{A} \quad (5)$$

Therefore,  $\widehat{V}_{21} = \frac{\partial^2 \widehat{V}}{\partial y \partial k} = -\frac{\partial \lambda}{\partial k} = \frac{-x\lambda f_{kL} + yf_k}{A} \geq 0$ . Hence we only need to prove that

$$(1 + \beta)\widehat{V}_{12} + \beta\widehat{V}_{11} + \widehat{V}_{22} < 0.$$

Indeed, it follows from (4), (5) that

$$\begin{aligned} (1 + \beta)\widehat{V}_{12} + \beta\widehat{V}_{11} + \widehat{V}_{22} &= \\ -(1 + \beta)\frac{\partial \lambda}{\partial k} + \beta \left[ \frac{\partial \lambda}{\partial k} f_k + \lambda(f_{kk} - f_{kL} \frac{\partial l^s}{\partial k}) \right] - \frac{\partial \lambda}{\partial y} \\ &= \frac{\beta \lambda f_{kk} A + \beta (\lambda f_{Lk})^2 \frac{1}{a} + x \lambda f_{kL} (1 - \beta)}{A}. \end{aligned}$$

It is easy to see that  $\frac{x\lambda f_{kL}(1-\beta)}{A} \leq 0$ .

Therefore, we now have to prove that

$$\begin{aligned} \frac{\beta\lambda f_{kk}A + \beta(\lambda f_{Lk})^2 \frac{1}{a}}{A} &< 0 \\ \Leftrightarrow f_{kk}A + \lambda(f_{Lk})^2 \frac{1}{a} &> 0. \end{aligned} \quad (6)$$

Note that  $\lambda = \frac{p}{f_{LL}}$ , (6) can be written as follows

$$f_{kk}f_{LL}(-A) > (f_{Lk})^2 \frac{p}{a}.$$

Since  $f$  is concave, we have

$$f_{kk}f_{LL} \geq (f_{Lk})^2.$$

Therefore, we just need to show that  $\frac{p}{a} < -A$ . Indeed, we have

$$\frac{p}{a} + A = \frac{p}{a} - \frac{y}{a} - x^2 = -\frac{ac - b^2}{a^2} - x^2 < 0.$$

Hence,

$$(1 + \beta)\widehat{V}_{12} + \beta\widehat{V}_{11} + \widehat{V}_{22} < 0.$$

■

The method to prove the differentiability of the policy function at a stationary optimal stock is based on Mitra (2000) where the derivative of the policy function equals the smaller characteristic root in absolute value associated with Euler equation (see also Mitra and Nishimura (2005)). Thus, we can state.

**Proposition 6** *Assume **U1-U4** and **F1-F4**. The policy function  $g(k)$  is differentiable at the unique nontrivial steady state  $(c^s, l^s, k^s)$  and  $g'(k) = |\lambda_1|$  where  $\lambda_1$  is the characteristic root of the Ramsey-Euler equations with the smaller absolute value.*

**Proof:** From Lemma 2, condition C1 of Proposition 7.3, Mitra (2000) is satisfied, and we can follow the result therein. ■

Assumption **U4** which is needed to obtain monotonicity of the optimal capital stock in the elastic labor supply framework can be dispensed with in the Mitra (2000) framework with inelastic labor supply.

## 5 Conclusion

This paper studies existence of equilibrium in the optimal growth model with elastic labor supply. This model is the workhorse of dynamic general equilibrium theory both endogenous and real business cycles. The results on existence of equilibrium have assumed strong conditions which are violated in some specifications of applied models.

*The usual method of proof that relies on defining prices to be equal to the discounting marginal utilities and then use the property that optimal programs satisfy Euler equations cannot apply directly in this case. To overcome the problem we provide an alternative argument that exploits the useful insights derived from the analysis of the Lagrange multipliers and Kuhn-Tucker optimality conditions for a infinite version of planner's problem.* This paper uses a separation argument to obtain Lagrange multipliers which lie in  $l^1$ . As the separation argument relies on convexity, strict convexity can be relaxed; this also means that assumptions on cross partials of utility functions are not needed (as in Coleman (1997), Greenwood and Huffman (1995) and Datta *et al* (2002)); and constant returns to scale is not needed. The representation theorem involves assumptions on asymptotic properties of the constraint set (which are weaker than Mackey continuity (see Bewley (1972) and Dechert (1982))). The assumptions ensure that the either the optimal sequence  $\{c_t, l_t\}_{t=0}^{\infty}$  is either always strictly interior or always equal to zero. Thus, one does not have to impose strong conditions, either Inada or  $\lim_{\epsilon \rightarrow 0} \frac{u(\epsilon, \epsilon)}{\epsilon} \rightarrow +\infty$  as in Le Van and Vailakis (2004) to ensure that the sequence of labor is strictly interior. This later condition is not satisfied, for example, in homogeneous period one utility functions. Thus, for a wide class of models, where assuming strict concavity of preferences or Inada conditions are not problematic, e.g., models with linear utility of leisure, one can still use approach of studying the social planner's problem knowing that these can be decentralized. In addition, one does not need to assume normality (or rule out backward bending labor supply curves) to study the competitive equilibrium. For further properties such as studying stability properties of the equilibrium, for nice characterizations, these are however, still needed.

## 6 Appendix

### Proof of Lemma1

**Proof:** i) Let  $k_0 > 0$  but assume that  $k_1^* = 0$ . Denote  $L_t^* = 1 - l_t^*$ . Since

$$c_0^* = f(k_0, L_0^*) > 0,$$

First, we claim that with  $l_1^* > 0$ .

Assume the contrary that  $l_1^* = 0$ . In this case, we prove that  $c_1^* > 0$ . Indeed, if  $c_1^* = 0$  then  $k_2^* = f(0, 1)$ . Choose  $\varepsilon > 0$  such that  $c_0^* > \varepsilon + \varepsilon^2$ . Let  $\alpha = \frac{\varepsilon+1}{\beta}$  and  $\gamma = \frac{\varepsilon+1}{\beta[c_0^* - (\varepsilon + \varepsilon^2)]}$ . Consider the alternative path  $(\mathbf{c}, \mathbf{l}, \mathbf{k})$  defined as follows:

- i)  $c_0 = c_0^* - (\varepsilon + \varepsilon^2)$ ,  $c_1 = c_0^*$
- ii)  $c_1 = \alpha\varepsilon$
- iii)  $l_0 = l_0^*$ ,  $l_1 = \gamma\varepsilon$
- iv)  $c_t = c_t^*$  and  $l_t = l_t^*$ ,  $\forall t \geq 2$
- v)  $k_1 = \varepsilon$ ,  $k_t = k_t^*$ ,  $\forall t \geq 2$ .

Observe that

$$\begin{aligned} c_0 + k_1 &= \\ c_0^* - (\varepsilon + \varepsilon^2) + \varepsilon &\leq c_0^* + k_1^* = f(k_0, L_0^*). \end{aligned}$$

Moreover,

$$\begin{aligned} &f(k_1, 1 - l_1) - k_2 - c_1 \\ &= f(\varepsilon, 1 - \gamma\varepsilon) - f(0, 1) - \alpha\varepsilon \\ &\geq \varepsilon[f_k(\varepsilon, 1 - \gamma\varepsilon) - f_L(\varepsilon, 1 - \gamma\varepsilon)\gamma - \alpha]. \end{aligned}$$

Due to the Inada conditions on  $F$ , the term inside the bracket is strictly positive for  $\varepsilon$  small enough. This proves feasibility of the alternative path.

Observe that as  $\varepsilon \rightarrow 0$  both  $\alpha$  and  $\gamma$  converge to a finite value. In addition,  $\frac{\alpha}{\gamma} = c_0$ . Define:

$$\begin{aligned} \Delta(\varepsilon) &= \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) - \sum_{t=0}^{\infty} \beta^t u(c_t^*, l_t^*) \\ &= [u(c_0, l_0) - u(c_0^*, l_0^*)] + \beta [u(c_1, l_1) - u(c_1^*, l_1^*)]. \end{aligned}$$

The concavity of  $u$  implies that

$$\begin{aligned}\Delta(\varepsilon) &= [u(c_1, l_1) - u(c_1^*, l_1^*)] + [u(c_0, l_0) - u(c_0^*, l_0^*)] \\ &\geq \beta u(\alpha\varepsilon, \gamma\varepsilon) - u_c(c_0, l_0)(\varepsilon + \varepsilon^2).\end{aligned}$$

If  $u_{cl} > 0$ , then

$$\begin{aligned}\Delta(\varepsilon) &\geq \beta u\left(\gamma\varepsilon\frac{\alpha\varepsilon}{\gamma\varepsilon}, \gamma\varepsilon\right) - u_c(c_0, l_0)(\varepsilon + \varepsilon^2) \\ &\geq \beta u\left(\frac{\alpha}{\gamma}, 1\right)\gamma\varepsilon - u_c(c_0, l_0)(\varepsilon + \varepsilon^2) \\ &\geq \beta u_c\left(\frac{\alpha}{\gamma}, 1\right)\alpha\varepsilon - u_c(c_0, l_0)(\varepsilon + \varepsilon^2) \\ &= \beta u_c(c_0, 1)\frac{\varepsilon^2 + \varepsilon}{\beta} - u_c(c_0, l_0)(\varepsilon + \varepsilon^2) \\ &= (\varepsilon^2 + \varepsilon)[u_c(c_0, 1) - u_c(c_0, l_0)] \geq 0.\end{aligned}$$

If  $u_{cl} \leq 0$ , then

$$\begin{aligned}\Delta(\varepsilon) &\geq \beta u(\alpha\varepsilon, \gamma\varepsilon) - u_c(c_0, l_0)(\varepsilon + \varepsilon^2) \\ &\geq \beta u_c(\alpha\varepsilon, \gamma\varepsilon)\alpha\varepsilon - u_c(c_0, l_0)(\varepsilon + \varepsilon^2) \\ &\geq (\varepsilon^2 + \varepsilon)[u_c(\alpha\varepsilon, 1) - u_c(c_0, l_0)].\end{aligned}$$

Due to the Inada conditions on  $u$ , the term inside the bracket becomes nonnegative for  $\varepsilon$  small enough. A contradiction.

Thus, we have  $c_1^* > 0$ . We claim that  $l_1^* > 0$ . Indeed, if this were false, define a feasible path as follows:

- i)  $l_1 = \varepsilon$ ,
- ii)  $c_1 = c_1^* + f(0, 1 - \varepsilon) - f(0, 1)$
- iii)  $c_t = c_t^*$ ,  $l_t = l_t^*$ ,  $\forall t \neq 1$ ,  $k_t = k_t^*$   $\forall t$ .

Define:

$$\begin{aligned}\Delta(\varepsilon) &= \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) - \sum_{t=0}^{\infty} \beta^t u(c_t^*, l_t^*) \\ &= \beta [u(c_1, \varepsilon) - u(c_1^*, 0)] \geq \beta u_c(c_1, \varepsilon)(f(\varepsilon, L_1^*) - f(0, L_1^*)) \\ &\quad + \beta u_l(c_1, \varepsilon)\varepsilon \geq \beta [-u_c(c_1, \varepsilon)(f_L(0, 1 - \varepsilon) + u_l(c_1, \varepsilon))\varepsilon].\end{aligned}$$

As  $\varepsilon \rightarrow 0$ ,  $u_l(c_1, \varepsilon) \rightarrow +\infty$  while  $-u_c(c_1, \varepsilon)(f_L(0, 1 - \varepsilon)) < +\infty$ . Hence, for  $\varepsilon > 0$  small enough,  $\Delta(\varepsilon) > 0$  : a contradiction. Thus,  $l_1^* > 0$ .

Now, we consider the alternative feasible path  $(\mathbf{c}, \mathbf{l}, \mathbf{k})$  defined as follows:

- i)  $c_0 = c_0^* - \varepsilon$ ,  $c_1 = c_1^* + f(\varepsilon, L_1^*) - f(0, L_1^*)$ ,  $c_t = c_t^*$ ,  $\forall t \geq 2$ ,
- ii)  $l_t = l_t^*, \forall t$
- iii)  $k_1 = \varepsilon$ ,  $k_t = k_t^*, \forall t \geq 2$ .

Define:

$$\Delta_\varepsilon = \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) - \sum_{t=0}^{\infty} \beta^t u(c_t^*, l_t^*)$$

The concavity of  $u$  and  $f$  implies that

$$\begin{aligned} \Delta(\varepsilon) &= u(c_0, l_0) - u(c_0^*, l_0^*) + \beta [u(c_1, l_1) - u(c_1^*, l_1)] \\ &\geq [-u_c(c_0, l_0) + \beta u_c(c_1, l_1) f_k(\varepsilon, L_1^*)] \varepsilon. \end{aligned}$$

As  $\varepsilon \rightarrow 0$ ,  $\beta u_c(c_1, l_1) f_k(\varepsilon, L_1^*) \rightarrow +\infty$  while  $u_c(c_0, l_0) \rightarrow u_c(c_0^*, l_0^*) < +\infty$ . Hence, for  $\varepsilon > 0$  small enough,  $\Delta(\varepsilon) > 0$  : a contradiction. It follows that  $k_1^* > 0$ . Working by induction we can show that  $k_t^* > 0$  for any  $t$ .

ii) It follows from proposition 10 in C. Le Van, M.H Nguyen and Y. Vailakis [2007] that there exists  $\gamma > 0$  such that  $k_t^* > \gamma \forall t$ . Suppose that there exist an optimal paths  $(\mathbf{c}^*, \mathbf{l}^*, \mathbf{k}^*)$  with  $c_0^* = 0$ , we can choose a feasible paths from this optimal paths where we just replace  $c_0^*, k_t^*$  with  $c_0 = \varepsilon_0 > 0$ ,  $k_t = k_t^* - \varepsilon_t$  in which  $\{\varepsilon_t\}$  is an increasing sequence bounded from above by  $\gamma$  ( for example,  $\varepsilon_t = \gamma - \frac{1}{t+n}$ ,  $n > 0$ ) such that  $c_t^* + k_{t+1}^* - \varepsilon_{t+1} \leq f(k_t^* - \varepsilon_t, L_t^*)$ . This feasible path create a new greater value than optimal value which leads to a contradiction. Thus  $c_t^* > 0$  for all  $t$ . It follows from (7) that  $l_t^* > 0$ . (Otherwise,  $\lambda_t^1$  would not belong to  $l_+^1$ ). ■

### Proof of Proposition 3

**Proof:** Consider Proposition2. It follow from Lemma 1 that  $c_t^* > 0, l_t^* > 0$ . Equation (3)-(5) in the Proposition2 imply that

$$\lambda_t^2 = \lambda_t^3 = \lambda_t^4 = 0.$$

Inada condition and equation (9) in the Proposition2 imply  $l_t^* < 1$ . Therefore  $\lambda_t^5 = 0$ .

Define a feasible path as follows:

- i)  $l_t = l_t^*, k_t = k_t^* \forall t$ ,
- ii)  $c_t = c_t^*, \forall t \neq T$ ,
- iii)  $c_T = c_T^* + \epsilon, \epsilon \in R$  such that  $c_T^* + \epsilon > 0$ .

Inequality (1) in the Proposition2 become

$$\beta^T u(c_T^*, l_T^*) - \lambda_T^1 c_T^* \geq \beta^T u(c_T^* + \epsilon, l_T^*) - \lambda_T^1 (c_T^* + \epsilon)$$

at any neighborhood of zero. Thus  $\lambda_T^1 = \beta^T u(c_T^*, l_T^*)$ .

Define

- i)  $l_t = l_t^*, c_t = c_t^* \forall t$ ,
- ii)  $k_t = k_t^*, \forall t \neq T + 1$ ,
- iii)  $k_{T+1} = k_{T+1}^* + \epsilon, \epsilon \in R$  such that  $k_{T+1}^* + \epsilon > 0$ .

Inequality (1) in the Proposition2 become

$$-\lambda_T^1 k_{T+1}^* + \lambda_{T+1}^1 f(k_{T+1}^*, L_{T+1}^*) \geq -\lambda_T^1 (k_{T+1}^* + \epsilon) + \lambda_{T+1}^1 f(k_{T+1}^* + \epsilon, L_{T+1}^*)$$

at any neighborhood of zero. Thus  $-\lambda_T^1 k_{T+1}^* + \lambda_{T+1}^1 f(k_{T+1}^*, L_{T+1}^*) = 0$ .

Replacing  $\lambda_T^1 = \beta^T u(c_T^*, l_T^*)$  into this equation we get Euler equation. ■

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