ROBERTS' THEOREM WITH NEUTRALITY: A SOCIAL WELFARE ORDERING APPROACH [*]

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PRELIMINARY VERSION - COMMENTS ARE WELCOME.

Abstract

We consider dominant strategy implementation in private values settings where the set of alternatives is finite, monetary transfers are allowed, and agents have quasilinear utilities. We show that any implementable and neutral social choice function must be a weighted welfare maximizer if its domain is open and connected. When the domain is unrestricted, Roberts' theorem with neutrality (Roberts, 1979) becomes a corollary to our result. Our proof technique uses a *social welfare ordering* approach, commonly used in aggregation literature in social choice theory.

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1 INTRODUCTION

The well known Gibbard-Satterthwaite (Gibbard, 1973; Satterthwaite, 1975) impossibility theorem in mechanism design asserts that, in unrestricted domains, every implementable social choice function which has at least three alternatives in its range must be dictatorial in the absence of transfers. However, scenarios where monetary transfers are necessary have increasingly become relevant. Auction-like settings assume that agents can entertain monetary transfers and the underlying utility function of every agent is quasi-linear in money. Execution of public projects often involve monetary transfers. Such settings with money and quasi-linear utility is the focus of this note. What kind of social choice functions can be implemented in dominant strategies in such settings?

Vickrey (1961); Clarke (1971); Groves (1973) showed that efficient social choice functions can be implemented by a unique family of transfer rules, now popularly known as Vickrey-Clarke-Groves (VCG) transfer schemes. Remarkably, when the domain is unrestricted (as in the Gibbard-Satterthwaite setup) and the range of the mechanism contains at least three alternatives, then the only (dominant strategy) implementable social choice functions are *affine maximizers*, i.e., slight generalizations of weighted efficiency. This result was proved by Roberts (1979) in his seminal paper. This result can be seen to be the parallel to the Gibbard-Satterthwaite theorem in quasi-linear utility environments.

As in the literature without money, the literature with quasi-linear utility has since tried to relax various assumptions in Roberts' theorem. Rochet (1987) shows that a certain *cycle monotonicity* property characterizes dominant strategy implementable social choice functions. Though this characterization is very general - works for any domains and any set of alternatives (finite or infinite) - it is not as useful as the Roberts' theorem since it does not give a functional form of the implementable social choice functions. Along the lines of Rochet (1987), Bikhchandani et al. (2006) and Saks and Yu (2005) have shown that a *weak monotonicity* property characterizes implementable social choice functions in auction settings when the set of alternatives is finite and the closure of the type space is convex [1]. Again, the precise functional form of the implementable social choice functions are missing in these characterizations. Hence, researchers continue to ponder the following question:

Under what subdomains can one derive a precise functional form of implementable social choice functions?

One of the central difficulties with answering this question is that the original proof of Roberts' theorem is long and complicated. Using almost the same structure and approach, Lavi et al. (2009) reduced the complexity of Roberts' original proof (see also Dobzinski and Nisan (2009)). Building on the technique of Lavi et al. (2009), Carbajal and Tourky (2009) extend Roberts' theorem to *continuous* domains. Other proofs of Roberts' theorem (for

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[1] See also Monderer (2008) for an elegant proof.

unrestricted domains) are Jehiel et al. (2008) and Vohra (2008).

1.1 OUR CONTRIBUTION

We focus on neutral social choice functions. Roberts' theorem with neutrality says that any implementable social choice function must be weighted welfare maximizer if the domain is unrestricted. We show that if the domain is open and connected then any implementable and neutral social choice function must be a weighted welfare maximizer. Roberts' theorem with neutrality becomes a corollary to our result.

Our proof is entirely novel. It differs from Roberts' original proof and the proofs in Lavi et al. (2009), Carbajal and Tourky (2009), Dobzinski and Nisan (2009), Jehiel et al. (2008) significantly. Our approach to proving Roberts' theorem with neutrality in open and connected domains can be summarized in three steps.

S1 We show that an implementable and neutral social choice function induces an ordering on the domain.

S2 This ordering satisfies three key properties: *weak Pareto, invariance, and continuity.*

S3 We then prove a result on representation of any ordering which satisfies these properties. For unrestricted domains, such a result is well known from Blackwell and Grishick (1954), d'Aspremont and Gevers (1977), Blackorby et al. (1984), Trockel (1992), and d'Aspremont and Gevers (2002).

Steps 1 and 2 are extremely simple and Step 3 is a classical result in social choice theory which we extend to open and connected domain. The result in Step 3 is of independent interest. **Need to write more about the approach**

1.2 A MOTIVATING EXAMPLE

Consider a city planner who wants to build a bridge across a river passing through the city. There are a finite number of locations where a bridge can be built. The set of possible allocations are thus the set of locations (note that not building the bridge is not a possible allocation). In this setting, it is natural to assume that the city planner has no preference on which bridge to build. So, he can be assumed to be neutral between the set of allocations. However, the citizens of the city have private utilities for different locations, which is represented by their type vectors over the set of allocations. Our results can be applied to a setting like this. We show that under some domain restrictions, the only implementable social choice functions are weighted welfare maximizers. However, our results do not extend to auction settings. The open domain condition is not satisfied in the auction settings. We elaborate on this fact later.

2 ROBERTS' AFFINE MAXIMIZER THEOREM

Let $A = \{a, b, c, \ldots\}$ be a finite set of alternatives or allocations. Suppose $|A| = m \geq 3$. Let $N = \{1, \ldots, n\}$ be a finite set of agents. The type of agent i is a vector in $\mathbb{R}^m$, where m is an integer greater than or equal to one. Denote by t_i the type (vector) of agent $i \in N$, where for every $a \in A$, t_i^a denotes the *value* of agent i for alternative a when his type is t_i. A type profile will be denoted by t, and consists of n vectors in $\mathbb{R}^m$. Alternately, one can view a type profile t to be an $n \times m$ matrix, where every row represents a type vector of an agent. The column vectors are vectors in $\mathbb{R}^n$. Let T_i be the set of all types (or type space) of agent i, i.e., every type t_i of agent i belongs to T_i. We assume $T_i = T$ for all $i \in N$. Let $\mathbb{T}$ be the set of all type profiles. We use the standard notation of t_{-i} to denote a type profile of agents in $N \setminus \{i\}$ and T_{-i} to denote the type spaces of agents in $N \setminus \{i\}$. Given a type profile t, we write t^a to denote the column vector corresponding to alternative a and t^{-a} to denote the column vectors in t except t^a. Given the set of type profiles $\mathbb{T}$, let $\mathbb{D}$ denote the set of all possible column vectors in the type profiles. For example, if $T = \mathbb{R}^m$, then $\mathbb{D} = \mathbb{R}^n$ and if $T = \mathbb{R}^m_+$, then $\mathbb{D} = \mathbb{R}^n_+$. In the sequel we will impose the following technical conditions on $\mathbb{D}$.

- We say $\mathbb{D}$ is **open from above** if for every $x \in \mathbb{D}$, there exists an $\epsilon \gg 0$ such that $(x + \epsilon) \in \mathbb{D}$.
- We say $\mathbb{D}$ is a **meet-semilattice** if for every $x, y \in \mathbb{D}$, $\min(x, y) \in \mathbb{D}$.

A social choice function is a mapping $f : \mathbb{T} \to A$. A payment function is a mapping $p : \mathbb{T} \to \mathbb{R}^n$. The payment of agent i at type profile t is denoted by $p_i(t)$.

DEFINITION **1** *A social choice function f is implementable (in dominant stragies) if there exists a payment function p such that for every $i \in N$ and every $t_{-i} \in \mathbb{T}_{-i}$ we have*

$$t_i^{f(t_i,t_{-i})} - p_i(t_i, t_{-i}) \geq t_i^{f(s_i,t_{-i})} - p_i(s_i, t_{-i}) \qquad \forall\ s_i, t_i \in T.$$

In this case, we say that p implements f.

Every social choice function satisfies certain properties if it is implementable. Below, we give one such useful property.

DEFINITION **2** *A social choice function f satisfies* **positive association of differences (PAD)** *if for every $s, t \in \mathbb{T}$ such that $f(t) = a$ with $s^a - t^a \gg s^b - t^b$ for all $b \neq a$, we have $f(s) = a$.*

LEMMA **1** **(Roberts (1979))** *Every implementable social choice function satisfies PAD.*

An intriguing question to ask is what social choice functions are implementable. In a remarkable result, Roberts (1979) characterized the set of all social choice functions when the type space is unrestricted and when the social choice function satisfies a condition called *non-imposition.*

DEFINITION **3** *A social choice function f satisfies* **non-imposition** *if for every $a \in A$, there exists $t \in \mathbb{T}$ such that $f(t) = a$.*

Using PAD and non-imposition, Roberts proved the following theorem.

THEOREM **1** **(Roberts (1979))** *Suppose $\mathbb{D} = \mathbb{R}^n$. If an implementable social choice function satisfies non-imposition, then there exists weights $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ and a deterministic real-valued function $\kappa : A \to \mathbb{R}$ such that for all $t \in \mathbb{T}$,*

$$f(t) \in \arg\max_{a \in A} \Big[\sum_{i \in N} \lambda_i t_i^a + \kappa(a)\Big]$$

This family of social choice functions are called **affine maximizer social choice functions**.

We will use the following stronger condition than non-imposition. Let ρ denote a permutation of set of alternatives A. For every type profile t, define $\gamma^\rho(t)$ to be a permutation of column vectors of t induced by the permutation ρ of A. Similarly, for any subset $S \subseteq A$, define $\eta^\rho(S)$ to be the permuted elements of S based on the permutation ρ.

DEFINITION **4** *A social choice function f is* **neutral** *if for every type profile $t \in \mathbb{T}$ and for all permutations ρ on A such that $t \neq \gamma^\rho(t)$, we have $\rho(f(t)) = f(\gamma^\rho(t))$.*

Neutrality requires that a social choice function does not discriminate between two social alternatives by their names. In many settings, this is a desirable condition. Non-imposition is implied by neutrality if $\mathbb{D} = \mathbb{R}^n$ or $\mathbb{R}^n_+$.

LEMMA **2** *Suppose f is an implementable and neutral social choice function and $\mathbb{D}$ is open from above. If f is neutral then it satisfies non-imposition.*

Proof: Fix an alternative $a \in A$. We will show that there exists a $t \in \mathbb{T}$ such that $f(t) = a$. Consider any arbitrary type profile s such that $f(s) = b \neq a$. Now, construct another type profile $u = (u^b = s^b + \epsilon, u^{-b} = s^{-b})$ for some $\epsilon \gg 0$. In particular, ϵ can be chosen such that $u^b \neq u^a$. Since $\mathbb{D}$ is open from above, $u^b \in \mathbb{D}$. By PAD, $f(u) = b$. Now, permute the columns a and b in u to get another type profile v. By neutrality, $f(v) = a$. Hence, f satisfies non-imposition. ■

Under neutrality, Roberts' theorem is modified as follows.

THEOREM **2** **(Roberts (1979))** *Suppose $\mathbb{D} = \mathbb{R}^n$. If an implementable social choice function satisfies neutrality, then there exists weights $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ such that for all $t \in \mathbb{T}$,*

$$f(t) \in \arg\max_{a \in A} \sum_{i \in N} \lambda_i t_i^a$$

The beauty of this theorem is that it gives a precise functional form of the neutral social choice functions that can be implemented. This family of social choice functions is called the **weighted welfare maximizer social choice functions**. If all the weights (λ_is) are equal in a weighted welfare maximizer social choice function, then we get the **efficient social choice function**.

Roberts' original proof is mainly geometric, and uses convexity and separating hyperplane arguments on an open set (see also a similar but simpler proof in Lavi et al. (2009)). We present an alternate approach to prove a weaker version of his theorem (Theorem 2), where we characterize the weighted welfare maximizer social choice functions. Our proof goes through three easy steps:

S1 First, we show that every neutral and implementable social choice function must induce an ordering of $\mathbb{D}$ if $\mathbb{D}$ is open from above and a meet-semilattice.

S2 We then show that this ordering must satisfy three properties: *weak Pareto, invariance, and continuity.*

S3 Finally, every ordering on $\mathbb{D}$ which satisfies these properties must come be a weighted welfare maximizer social choice function if $\mathbb{D}$ is open and connected.

3 SOCIAL WELFARE ORDERING

Given a social choice function f we define the following set. For every $t \in \mathbb{T}$, the **choice set** at t is defined as:

$$C^f(t) = \{a \in A : \forall \epsilon \gg 0 \text{ and } \forall (t^a + \epsilon, t^{-a}) \in \mathbb{T}, f(t^a + \epsilon, t^{-a}) = a\}.$$

We first show that choice sets are always non-empty.

LEMMA **3** *Let f be an implementable social choice function and $\mathbb{D}$ be open from above. Then, for every $t \in \mathbb{T}$, $f(t) \in C^f(t)$.*

Proof: Consider $t \in \mathbb{T}$, and let $f(t) = a$. Let $s = (s^a = t^a + \epsilon, s^{-a} = t^{-a})$ for some $\epsilon \gg 0$ such that $s \in \mathbb{T}$. Since $\mathbb{D}$, such an ϵ exists. By PAD, $f(s) = a$. ■

Now, we prove an important lemma for our proof.

LEMMA **4** *Let f be an implementable social choice function and $\mathbb{D}$ be open from above and a meet-semilattice. Consider two type profiles $t = (t^a, t^b, t^{-ab})$, $s = (s^a = t^a, s^b = t^b, s^{-ab})$.*

a) Suppose $a, b \in C^f(t)$. Then, either $\{a, b\} \subseteq C^f(s)$ or $\{a, b\} \cap C^f(s) = \emptyset$.

b) Suppose $a \in C^f(t)$ but $b \notin C^f(t)$. Then $b \notin C^f(s)$.

Proof: Suppose $a, b \in C^f(t)$. Now, consider a type profile $u = (u^a = t^a, u^b = t^b, u^{-ab})$, where $u_i^c = \min(t_i^c, s_i^c)$ for all $i \in M$ and for all $c \notin \{a, b\}$. Since $\mathbb{D}$ is a meet-semilattice, $u \in \mathbb{D}$.

a) Suppose $a, b \in C^f(t)$. We will first show that $a, b \in C^f(u)$. Choose an $\epsilon \gg 0$. Since $a \in C^f(t)$, we know that $f(t^a + \frac{\epsilon}{2}, t^b, t^{-ab}) = a$. By PAD, $f(t^a + \epsilon, t^b, u^{-ab}) = a$. Hence, $a \in C^f(u)$. Using an analogous argument, $b \in C^f(u)$.

   Now, suppose that $a \in C^f(s)$ and assume for contradiction $b \notin C^f(s)$. Choose an $\epsilon \gg 0$ and arbitrarily close to zero. We show that $f(t^a + 2\epsilon, t^b + 3\epsilon, s^{-ab}) \neq b$. Assume for contradiction, $f(t^a + 2\epsilon, t^b + 3\epsilon, s^{-ab}) = b$. By PAD, $f(t^a, t^b + 4\epsilon, s^{-ab}) = b$. Since ϵ can be made arbitrarily small, this implies that $b \in C^f(s)$. This is a contradiction.

   Next, we show that $f(t^a + 2\epsilon, t^b + 3\epsilon, s^{-ab}) \neq c$ for any $c \notin \{a, b\}$. Assume for contradiction $f(t^a + 2\epsilon, t^b + 3\epsilon, s^{-ab}) = c$ for some $c \notin \{a, b\}$. By PAD, $f(t^a + 2\epsilon, t^b, s^c + \frac{\epsilon}{2}, s^{-abc}) = c$. Also, since $a \in C^f(s)$, we know that $f(t^a + \epsilon, t^b, s^c, s^{-abc}) = a$. By PAD, $f(t^a + 2\epsilon, t^b, s^c + \frac{\epsilon}{2}, s^{-abc}) = a$. This is a contradiction.

   Hence, $f(t^a + 2\epsilon, t^b + 3\epsilon, s^{-ab}) = a$. By PAD, $f(t^a + \frac{5\epsilon}{2}, t^b + 3\epsilon, u^{-ab}) = a$. We show that $f(t^a, t^b + \epsilon', u^{-ab}) \neq b$ for all $0 \ll \epsilon' \ll \frac{\epsilon}{2}$. Assume for contradiction $f(t^a, t^b + \epsilon', u^{-ab}) = b$ for some $0 \ll \epsilon' \ll \frac{\epsilon}{2}$. By PAD, $f(t^a + \frac{5\epsilon}{2}, t^b + 3\epsilon, u^{-ab}) = b$. This is a contradiction. Hence, $f(t^a, t^b + \epsilon', u^{-ab}) \neq b$ for some $\epsilon' \gg 0$. This implies that $b \notin C^f(u)$, which is a contradiction. Hence, $a \in C^f(s)$ implies that $b \in C^f(s)$.

   Now, suppose that $a \notin C^f(s)$. Assume for contradiction $b \in C^f(s)$. Exchanging the role of a and b above, we get that $a \in C^f(s)$. This is a contradiction. Hence, if $a \notin C^f(s)$ then $b \notin C^f(s)$. This implies that either $\{a, b\} \subseteq C^f(s)$ or $\{a, b\} \cap C^f(s) = \emptyset$.

b) As in part (a), $a \in C^f(u)$. Now, assume for contradiction, $b \in C^f(s)$. If $a \notin C^f(s)$, then exchanging the role of a and b in the second half of (a), we get that $a \notin C^f(u)$. This is a contradiction. If $a \in C^f(s)$, then we have $a, b \in C^f(s)$ but $a \in C^f(t)$. By part (a), $b \in C^f(t)$. This is a contradiction.

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We will define an ordering on $\mathbb{D}$ induced by an implementable social choice function. In general, we will talk about any ordering R on $\mathbb{D}$. The symmetric component of an ordering R will be denoted as I and the anti-symmetric component will be denoted as P. Note that a social choice function f is a mapping $f : \mathbb{T} \to A$. Hence, for every type profile t, a social choice function can be thought to be picking a column vector (which belongs to $\mathbb{D}$) in t. We intend to show that in the process of picking these column vectors in $\mathbb{D}$ in an implementable manner, a social choice function induces a social welfare ordering under neutrality. The next lemma shows that if f is neutral then the choice set is also "neutral".

LEMMA **5** *Suppose f is an implementable and neutral social choice function and $\mathbb{D}$ is open from above. Then, for any permutation ρ of A we have $\eta^{\rho}(C^f(t)) = C^f(\gamma^{\rho}(t))$ for all $t \in \mathbb{T}$ such that $t \neq \gamma^{\rho}(t)$.*

Proof: Fix a type profile t and a permutation ρ of A such that $t \neq \gamma^{\rho}(t)$. Consider $a \in C^f(t)$. Denote $u = (u^a = t^a + \epsilon, u^{-a} = t^{-a})$ for some $\epsilon \gg 0$ such that $u \neq \gamma^{\rho}(u)$ (this is possible since $t \neq \gamma^{\rho}(t)$) . By definition of the choice set, $f(u) = a$. By neutrality, $f(\gamma^{\rho}(u)) = \rho(f(u))$. Hence, $\rho(a) \in C^f(\gamma^{\rho}(t)$.

Similarly, suppose $b \notin C^f(t)$ and let $v = (v^b = t^b + \epsilon, v^{-b} = t^{-b})$ for some $\epsilon \gg 0$ such that $v \neq \gamma^{\rho}(v)$ (this is possible since $t \neq \gamma^{\rho}(t)$). By definition of the choice set, $f(v) = a \neq b$. By neutrality $f(\gamma^{\rho}(v)) = \rho(a) \neq \rho(b)$. Hence, $\rho(b) \notin C^f(\gamma^{\rho}(t))$. This implies that $\eta^{\rho}(C^f(t)) = C^f(\gamma^{\rho}(t))$. ■

The following is a useful lemma that we will use in the proofs.

LEMMA **6** *Suppose f is an implementable and neutral social choice function and $\mathbb{D}$ is open from above. Consider a type profile $t \in \mathbb{D}$ such that $t^a = t^b$ for some $a, b \in A$. If $a \in C^f(t)$ then $b \in C^f(t)$ and if $a \notin C^f(t)$ then $b \notin C^f(t)$.*

Proof: Consider a profile $u = (u^a = t^a + \epsilon, u^{-a} = t^{-a})$ for some $\epsilon \gg 0$. Since $a \in C^f(t)$, $f(u) = a$. Now permute columns a and b in u to get another profile $v = (v^a = t^a = t^b, v^b = t^a + \epsilon = t^b + \epsilon, v^{-ab} = t^{-ab})$. Clearly, $v \neq u$. By neutrality, $f(v) = b$. Hence, $b \in C^f(t)$.

For the second part, assume for contradiction, $a \notin C^f(t)$ but $b \in C^f(t)$. Exchanging the role of a and b in the first part, we get that $b \in C^f(t)$ implies $a \in C^f(t)$. This is a contradiction. ■

DEFINITION **5** *A* **social welfare ordering** *R^f induced by a social choice function f is a relation on $\mathbb{D}$ defined as follows. The symmetric component of R^f is denoted by I^f and the antisymmetric component of R^f is denoted by P^f. Pick $x, y \in \mathbb{D}$.*

We say xP^fy if and only if there exists a profile t with $t^a = x$ and $t^b = y$ for some $a, b \in A$ such that $a \in C^f(t)$ but $b \notin C^f(t)$.

We say xI^fy if and only if there exists a profile t with $t^a = x$ and $t^b = y$ for some $a, b \in A$ such that $a, b \in C^f(t)$.

PROPOSITION **1** *Suppose f is an implementable and neutral social choice function and $\mathbb{D}$ is open from above and a meet-semilattice. Then, the relation R^f induced by f on $\mathbb{D}$ is an ordering.*

Proof: We first show that R^f is well-defined. Pick $x, y \in \mathbb{D}$. We consider two cases.

CASE 1: Suppose xP^fy. Then there exists a type profile t and some $a, b \in A$ such that $a \in C^f(t)$ but $b \notin C^f(t)$. Consider any other type profile s such that $s^a = x$ and $s^b = y$. By

Lemma 4, if $a \in C^f(s)$, then $b \notin C^f(s)$. Consider any other profile u and $(c,d) \neq (a,b)$ such that $u^c = x$ and $u^d = y$ but $(u^a, u^b) \neq (x,y)$. We can permute u to get another profile v such that $v^a = x$ and $v^b = y$. By Lemma 4, $a \in C^f(v)$ but $b \notin C^f(v)$. By Lemma 5, $c \in C^f(u)$ and $d \notin C^f(u)$. Hence, the choice of a and b is without loss of generality. So, P^f is well defined.

Case 2: Suppose xI^fy. Then there exists a type profile t and some $a, b \in A$ such that $a, b \in C^f(t)$. Consider any other type profile s such that $s^a = x$ and $s^b = y$. By Lemma 4, if $a \in C^f(s)$, then $b \in C^f(s)$ and if $a \notin C^f(s)$ then $b \notin C^f(s)$. By Lemma 5 (as in Case 1), the choice of a and b is without loss of generality. This shows that I^f is well defined.

We next show that R^f is reflexive. Consider $x \in \mathbb{D}$ and the profile where $t^a = x$ for all $a \in A$. By Lemma 6, $C^f(t) = A$. Hence, xI^fx.

Next, we show that R^f is complete. Choose $x, y \in \mathbb{D}$. Consider a type profile t where each column vector is either x or y with at least one column vector having x and at least one column vector having y. If $f(t) = a$, then either $t^a = x$ or $t^a = y$. Without loss of generality, let $f(t) = a$ and $t^a = x$. By Lemma 6, there are two cases to consider.

Case 1: For all b with $t^b = y$ we have $b \in C^f(t)$. Hence, xI^fy.

Case 2: For all b with $t^b = y$ we have $b \notin C^f(t)$. Then, we get xP^fy.

This completes the argument that R^f is complete, and hence, a binary relation. Now, we prove that R^f is transitive. Consider $x, y, z \in \mathbb{D}$. Consider a type profile t, where each column has value in $\{x, y, z\}$ with at least one column having value x, at least one column having value y, and at least one column having value z (this is possible since $|A| = m \geq 3$).

Due to Lemma 4 and 5, without loss of generality let $t^a = x, t^b = y, t^c = z$. We prove transitivity of P^f and I^f, and this implies transitivity of R^f.

Transitivity of P^f: Suppose xP^fy and yP^fz. This implies that $a \in C^f(t)$ but $b \notin C^f(t)$. Since yP^fz, we get that $c \notin C^f(t)$. Since $c \notin C^f(t)$, we have xP^fz.

Transitivity of I^f: Suppose xI^fy and yI^fz. This implies that $a, b \in C^f(t)$. But yI^fz implies that $c \in C^f(t)$ too. This implies that xI^fz.

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4 PROPERTIES OF SOCIAL WELFARE ORDERING

In this section, we fix an implementable neutral social choice function f. Then, we go on to prove that the social welfare ordering R^f defined in the last section satisfies three desirable properties.

DEFINITION **6** *An ordering R on $\mathbb{D}$ satisfies* **weak Pareto** *if for all $x, y \in \mathbb{D}$ with $x \gg y$ we have xPy.*

DEFINITION **7** *An ordering R on $\mathbb{D}$ satisfies* **invariance** *if for all $x, y \in \mathbb{D}$ and all $z \in \mathbb{R}^n$ such that $(x+z), (y+z) \in \mathbb{D}$ we have xPy implies $(x+z)P(y+z)$ and xIy implies $(x+z)I(y+z)$.*

DEFINITION **8** *An ordering R on $\mathbb{D}$ satisfies* **continuity** *if for all $x \in \mathbb{D}$, the sets $U^x = \{y \in \mathbb{D} : yRx\}$ and $L^x = \{y \in \mathbb{D} : xRy\}$ are closed in $\mathbb{D}$.*

PROPOSITION **2** *Suppose f is an implementable and neutral social choice function and $\mathbb{D}$ is open from above and a meet-semilattice. Then the social welfare ordering R^f induced by f on $\mathbb{D}$ satisfies weak Pareto, invariance, and continuity.*

Proof: The proof goes in three steps, showing that R^f satisfies each of the properties.

WEAK PARETO: Choose $x, y \in \mathbb{D}$ such that $x \gg y$. Start with a profile t where $t^a = y$ for all $a \in A$. Suppose $f(t) = b$. Consider another profile $s = (s^b = x, s^{-b} = t^{-b})$ (i.e. column vector corresponding to b is changed from y to x). By PAD, $f(s) = b$ and hence $b \in C^f(s)$. We show that for any $a \neq b$ we have $a \notin C^f(s)$. Choose $\epsilon \gg 0$ but $\epsilon \ll x - y$. By PAD, $f(t^a + \epsilon, s^b = x, t^{-ab}) = b$. Hence, $a \notin C^f(s)$. This shows that $b \in C^f(s)$ but $a \notin C^f(s)$. Hence, by Proposition 1, $xP^f y$.

INVARIANCE: Choose $x, y \in \mathbb{D}$ and $z \in \mathbb{R}^n$ such that $(x+z), (y+z) \in \mathbb{D}$. We consider two cases.

CASE 1: Suppose $xP^f y$. We show that $(x+z)P^f(y+z)$. Since $xP^f y$, there exists a profile $t = (t^a = x, t^b = y, t^{-ab})$ such that $a \in C^f(t)$ but $b \notin C^f(t)$. Consider the profile s, where $s^c = t^c + z$ for all $c \in A$. Fix $\epsilon \gg 0$. Since $a \in C^f(t)$, $f(t^a + \frac{\epsilon}{2}, t^b, t^{-ab}) = a$. Hence, by PAD $f(s^a + \epsilon, s^b, s^{-ab}) = a$. This shows that $a \in C^f(s)$. Since $b \notin C^f(t)$, there is some $\epsilon \gg 0$ such that $f(t^a, t^b + \epsilon, t^{-ab}) \neq b$. We show that $f(s^a, s^b + \frac{\epsilon}{2}, s^{-ab}) \neq b$. Assume for contradiction $f(s^a, s^b + \frac{\epsilon}{2}, s^{-ab}) = b$. By PAD, $f(t^a, t^b + \epsilon, t^{-ab}) = b$. This is a contradiction. Hence, $f(s^a, s^b + \frac{\epsilon}{2}, s^{-ab}) \neq b$. This implies that $b \notin C^f(s)$. Using Proposition 1, $(x+z)P^f(y+z)$.

CASE 2: Suppose $xI^f y$. We show that $(x+z)I^f(y+z)$. Then, there exists a profile $t = (t^a = x, t^b = y, t^{-ab})$ such that $a, b \in C^f(t)$. Consider the profile s, where $s^c = t^c + z$

for all $c \in A$. Fix $\epsilon \gg 0$. Since $a \in C^f(t)$, $f(t^a + \frac{\epsilon}{2}, t^b, t^{-ab}) = a$. Hence, by PAD $f(s^a + \epsilon, s^b, s^{-ab}) = a$. This shows that $a \in C^f(s)$. Using an analogous argument, $b \in C^f(s)$. Hence, by Proposition 1, $(x + z)I^f(y + z)$.

Continuity: Fix $x \in \mathbb{D}$. We show that the set $U^x = \{y \in \mathbb{D} : yR^f x\}$ is closed. Take an infinite sequence $y_1, y_2, \ldots$ such that every point y_n in this sequence satisfies $y_n R^f x$. Let this sequence converge to z. Assume for contradiction $xP^f z$. Consider a type profile t such that $t^a = x$ and $t^c = z$ for all $c \neq a$. Since $xP^f z$, we have $c \notin C^f(t)$ for all $c \neq a$. Hence, $C^f(t) = \{a\}$.

Consider $b \neq a$. Since $b \notin C^f(t)$, we know that there exists $\epsilon \gg 0$ and ϵ arbitrarily close to the zero vector such that $f(t^a, t^b + \epsilon, t^{-ab}) \neq b$. We show that $f(t^a, t^b + \epsilon, t^{-ab}) \neq c$ for all $c \notin \{a, b\}$. Assume for contradiction $f(t^a, t^b + \epsilon, t^c, t^{-abc}) = c$ for some $c \notin \{a, b\}$. Then, by PAD, $f(t^a, t^b, t^c + \epsilon'', t^{-abc}) = c$ for all $\epsilon'' \gg 0$. This implies that $c \in C^f(t)$, which is a contradiction. Hence, $f(t^a, t^b + \epsilon, t^{-ab}) = a$.

Now, choose $\epsilon' \gg 0$ such that $\epsilon \gg \epsilon'$. Define $s = (s^a = t^a, s^b = t^b + \epsilon', s^{-ab} = t^{-ab})$. We show that $b \notin C^f(s)$ and $a \in C^f(s)$. Assume for contradiction $b \in C^f(s)$. Then, $f(t^a, t^b + \epsilon, t^{-ab}) = b$ since $\epsilon \gg \epsilon'$. But this is a contradiction since $f(t^a, t^b + \epsilon, t^{-ab}) = a$. Also, since $f(t^a, t^b + \epsilon, t^{-ab}) = a$ and $\epsilon' \ll \epsilon$, by PAD, $f(t^a + \epsilon'', t^b + \epsilon', t^{-ab}) = a$ for all $\epsilon'' \gg 0$. Hence, $a \in C^f(s)$. So, we have $a \in C^f(s)$ but $b \notin C^f(s)$. By Proposition 1, $xP^f(z + \epsilon')$.

Since the sequence converges to z, there is a point $z' \in \mathbb{D}$ arbitrarily close to z such that $z'R^f x$. Since z is arbitrarily close to z', by weak Pareto, $(z + \epsilon')P^f z'$. Using $z'R^f x$, we get $(z + \epsilon')P^f x$. This is a contradiction to the fact that $xP^f(z + \epsilon')$.

To show $L^x = \{y \in \mathbb{D} : xR^f y\}$ is closed, take an infinite sequence $y_1, y_2, \ldots$ such that every point y_n in this sequence satisfies $xR^f y_n$. Let this sequence converge to z. Assume for contradiction $zP^f x$. Interchanging the role of x and z in the previous argument, we will get $zP^f(x + \epsilon')$ for some $\epsilon' \gg 0$. Since the sequence converges to z, there is a point $z' \in \mathbb{D}$ arbitrarily close to z such that $xR^f z'$. Since z is arbitrarily close to z, $(x + \epsilon')P^f z$ by weak Pareto. This is a contradiction to the fact that $zP^f(x + \epsilon')$. ■

5 Open and Connected Domains

In this section, we prove the main result. To prove this result, we prove the following result when $\mathbb{D}$ is convex and open. When $\mathbb{D} = \mathbb{R}^n$, this result is well known since Blackwell and Grishick (1954). Recent proofs of this case can be found in the utility representation literature - d'Aspremont and Gevers (1977), Blackorby et al. (1984), Trockel (1992), and d'Aspremont and Gevers (2002).

Proposition **3** *Suppose an ordering R on $\mathbb{D}$ satisfies weak Pareto, invariance, and conti-*

nuity. If $\mathbb{D}$ is convex and open, then there exists weights $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ and for all $x, y \in \mathbb{R}^n$

$$xRy \Leftrightarrow \sum_{i \in N} \lambda_i x_i \geq \sum_{i \in N} \lambda_i y_i.$$

Proof: Fix any $z \in \mathbb{D}$. Denote $U^z = \{x : xRz\}$, $L^z = \{x : zRx\}$, $\mathbb{D} \setminus L^z = \{x : xPz\}$, and $\mathbb{D} \setminus U^z = \{x : zPx\}$.

STEP 1: We first show that the sets $U^z, L^z, \mathbb{D} \setminus U^z$, and $\mathbb{D} \setminus L^z$ are convex. We make use of the following fact here.

FACT **1** *Consider a set $X \subseteq \mathbb{D}$ and let X satisfy the property that if $x, y \in X$ then $\frac{x+y}{2} \in X$. If X is open in $\mathbb{D}$ or closed in $\mathbb{D}$, then X is convex.*

The proof of this fact is given in the appendix. By continuity, each of the sets $U^z, L^z, \mathbb{D} \setminus U^z$, and $\mathbb{D} \setminus L^z$ are either open or closed in $\mathbb{D}$. Hence, by Fact 1, we only need to verify that these sets are closed under the midpoint operation.

Consider U^z. Now, let $x, y \in \mathbb{D}$ such that xRz and yRz. We will show that $\frac{x+y}{2}Rz$. Note that $\frac{x+y}{2} \in \mathbb{D}$ because $\mathbb{D}$ is convex. Now, assume for contradiction that $zP\frac{x+y}{2}$. This implies that $xP\frac{x+y}{2}$ and $yP\frac{x+y}{2}$. By invariance, $x + \frac{y-x}{2}P\frac{x+y}{2} + \frac{y-x}{2}$. Hence, $\frac{x+y}{2}Py$. This is a contradiction. Hence, the set U^z is convex.

Similar arguments show that L^z, $\mathbb{D} \setminus L^z$, and $\mathbb{D} \setminus U^z$ are convex.

STEP 2: We now show that z is a boundary point of U^z. Let $B_\delta(z) = \{x : \|x - z\| < \delta\}$, where $\delta \in \mathbb{R}_+$. Since $\mathbb{D}$ is open, there exists $\epsilon \gg 0$ such that $(z + \epsilon) \in \mathbb{D} \cap B_\delta(z)$ and, by weak Pareto, $(z + \epsilon)Pz$. Further, since $\mathbb{D}$ is open, ϵ can be chosen such that $(z - \epsilon) \in \mathbb{D} \cap B_\delta(z)$, and by weak Pareto, $zP(z - \epsilon)$. Hence, for every $\delta > 0$, there exists a point in $B_\delta(z)$ which is in U^z and another point which is not in U^z. This shows that z is a boundary point of U^z.

STEP 3: By the supporting hyperplane theorem, there exists a hyperplane through z supporting the set U^z, i.e., there exists a non-zero vector $\lambda \in \mathbb{R}^n \setminus \{0\}$ such that for all $x \in U^z$,

$$\sum_{i=1}^{n} \lambda_i x_i \geq \sum_{i=1}^{n} \lambda_i z_i.$$

Denote the intersection of this hyperplane with the set $\mathbb{D}$ as H^z.

STEP 4: We next show that $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$. Assume for contradiction $\lambda_j < 0$ for some $j \in N$. Since $\mathbb{D}$ is open there exists $\epsilon \gg 0$ such that $(z + \epsilon) \in \mathbb{D}$. Moreover, we can choose ϵ such that

$$\sum_{i=1}^{n} \lambda_i \epsilon_i < 0.$$

By weak Pareto $(z+\epsilon)Pz$. Hence, $(z+\epsilon) \in U^z$. Thus,

$$\sum_{i=1}^{n} \lambda_i(z_i + \epsilon_i) \geq \sum_{i=1}^{n} \lambda_i z_i.$$

This implies that

$$\sum_{i=1}^{n} \lambda_i \epsilon_i \geq 0.$$

This is a contradiction. Hence, $\lambda_i \geq 0$ for all $i \in N$.

Step 5: Now, consider $x \in \mathbb{D}$ such that

$$\sum_{i=1}^{n} \lambda_i x_i > \sum_{i=1}^{n} \lambda_i z_i.$$

We will show that xPz. Assume for contradiction zRx. We consider two cases.

Case 1: Suppose zPx. Since $\mathbb{D}$ is open, there exists a point z' in $B_\delta(z)$ for some $\delta \in \mathbb{R}_+$ such that

a) z lies on the line segment joining z' and x and

b) x and z' lies on opposite sides of the hyperplane H_z, i.e.,

$$\sum_{i=1}^{n} \lambda_i z'_i < \sum_{i=1}^{n} \lambda_i z_i.$$

By (b) and using Step 3, zPz'. By our assumption zPx. Hence, $x, z' \in \mathbb{D} \setminus U^z$. By Step 1, $\mathbb{D} \setminus U^z$ is convex. Since z is in the convex hull of x and z', we get that zPz. This is a contradiction.

Case 2: Suppose zIx. Since $\mathbb{D}$ is open, there exists $x' = x - \epsilon$ for some $\epsilon \gg 0$ such that

$$\sum_{i=1}^{n} \lambda_i x'_i > \sum_{i=1}^{n} \lambda_i z_i.$$

By weak Pareto xPx'. Hence, zPx'. By Case 1, this is not possible. This is a contradiction.

Hence, in both cases we reach a contradiction, and conclude that xPz.

Step 6: Now, consider $x \in \mathbb{D}$ such that

$$\sum_{i=1}^{n} \lambda_i x_i = \sum_{i=1}^{n} \lambda_i z_i.$$

We will show that xIz. Suppose not. There are two cases to consider.

Case 1: Assume for contradiction xPz. By continuity, the set $\{y : yPz\}$ is open in $\mathbb{D}$. Since $\mathbb{D}$ is open in $\mathbb{R}^n$, we get that $\{y : yPz\}$ is open in $\mathbb{R}^n$. Hence, there exists $\delta \in \mathbb{R}_+$ such that for every point in $x' \in B_\delta(x)$ we have $x'Pz$. Choose $\epsilon \gg 0$ such that for $x'' = x - \epsilon$ we have $x'' \in B_\delta(x)$. Hence, $x''Pz$. By Step 4, $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$. Hence, we get

$$\sum_{i=1}^n \lambda_i x''_i < \sum_{i=1}^n \lambda_i z_i.$$

But this is a contradiction since $x''Pz$ implies $x'' \in U^z$, which in turn implies that

$$\sum_{i=1}^n \lambda_i x''_i \geq \sum_{i=1}^n \lambda_i z_i.$$

Case 2: Assume for contradiction zPx. By continuity, the set $\{y : zPy\}$ is open in $\mathbb{D}$. Hence, there exists $\delta \in \mathbb{R}_+$ such that for every point in $x' \in B_\delta(x)$ we have zPx'. Choose $\epsilon \gg 0$ such that for $x'' = x + \epsilon$ we have $x'' \in B_\delta(x)$. Hence, zPx''. By Step 4, $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$. Hence, we get

$$\sum_{i=1}^n \lambda_i x''_i > \sum_{i=1}^n \lambda_i z_i.$$

By Step 5, this implies that $x''Pz$. This is a contradiction.

This shows that for any z, there exists $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ such that for all $x \in \mathbb{D}$, we have

$$xRz \Leftrightarrow \sum_{i=1}^n \lambda_i x_i \geq \sum_{i=1}^n \lambda_i z_i.$$

In other words, H^z contains all the points in $\mathbb{D}$ which are indifferent to z under R. Moreover, on one side of H^z we have points in $\mathbb{D}$ which are better than z under R and on the other side, we have points which are worse than z under R.

Finally, pick any two points x and y in $\mathbb{D}$. By weak Pareto and invariance, the hyperplanes H^x and H^y are parallel to each other. Hence, the proposition is proved. ■

We are now ready to state our main result.

Theorem **3** *Suppose f is an implementable and neutral social choice function and $\mathbb{D} \subseteq \mathbb{R}^n$ is open and connected. Then there exists weights $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ such that for all $t \in \mathbb{T}$,*

$$f(t) \in \arg\max_{a \in A} \sum_{i \in N} \lambda_i t^a_i.$$

Proof: Define the n-dimensional open square with sides of length δ and centered at $z \in \mathbb{R}^n$ as $\Gamma_\delta(z) = \{x \in \mathbb{R}^n : \max_{i \in N} |x_i - z_i| < \frac{\delta}{2}\}$. It is well known that Γ_δ forms the basis of the topology on $\mathbb{R}^n$ (and hence on $\mathbb{D}$) induced by the square metric and every open set can be written as a collection of all unions of basis elements (see Munkres (2008), for example). Moreover, since $\mathbb{D}$ is connected, we can place the squares in such a manner that every square intersects with some other square and they are connected (two squares are connected if there is a path from every point in one square to every point in the other square). This is true since $\mathbb{D}$ is open and connected, and hence, it is path connected.

Now, choose $z \in \mathbb{D}$, and consider $\Gamma_\delta(z)$ for some $\delta > 0$ such that $\Gamma_\delta(z) \subseteq \mathbb{D}$. Now, $\Gamma_\delta(z)$ is open (and hence, open from above), convex, and a meet-semilattice. Hence, by Proposition 1, a neutral and implementable SCF f induces a social welfare ordering R^f on $\Gamma_\delta(z)$. By Proposition 2, R^f satisfies continuity, weak Pareto, and invariance. By proposition 3, there exists weights $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ such that for every $x, y \in \Gamma_\delta(z)$ we have

$$xR^f y \Leftrightarrow \sum_{i \in N} \lambda_i x_i \geq \sum_{i \in N} \lambda_i y_i.$$

Now, since the squares that fill up $\mathbb{D}$ intersect each other and are connected, the weights λs defined by ordering R^f in each of these squares must be the same. Hence, we get that R^f is an ordering on $\mathbb{D}$ and for every $x, y \in \mathbb{D}$ we have

$$xR^f y \Leftrightarrow \sum_{i \in N} \lambda_i x_i \geq \sum_{i \in N} \lambda_i y_i.$$

Finally, by Lemma 3 for all $t \in \mathbb{D}^m$, $f(t) \in C^f(t)$. Hence, $t^{f(t)} R^f t^b$ for all $b \in A$ and for all $t \in \mathbb{D}^m$. This proves the theorem. ■

5.1 REMARKS

OPEN AND CONNECTED $\mathbb{T}$. Note that if $\mathbb{D}$ is connected and open, $\mathbb{D}^m$ is also connected and open. Further, if $\mathbb{D}^m = \mathbb{T}^n$ is connected and open, $\mathbb{T}$ is also connected and open. Hence, we can equivalently say $\mathbb{T}$ is open and connected in Theorem 3.

WEIGHTED VCG MECHANISMS. It is well known that if f is a weighted welfare maximizer with weights $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$, then the following payment function $p : \mathbb{T}^n \to \mathbb{R}^n$ makes the social choice function strategy-proof. For all $i \in N$ with $\lambda_i = 0$, $p_i(t) = 0$ for all $t \in \mathbb{T}^n$. For all $i \in N$ with $\lambda_i > 0$,

$$p_i(t) = \frac{1}{\lambda_i}\Big[\sum_{j \neq i} \lambda_j t_j^{f(t)}\Big] + h_i(t_{-i}) \qquad \forall\, t \in \mathbb{T}^n.$$

where $h_i : \mathbb{T}^{n-1} \to \mathbb{R}$. Due to the assumption in Theorem 3, $\mathbb{T}$ is connected, and hence, revenue equivalence holds in this setting (Chung and Olszewski, 2007; Heydenreich et al., 2009). Hence, these are the *only* payment functions which makes f strategy-proof. Payments of this form are called the weighted Vickrey-Clarke-Groves (VCG) payments, and the corresponding mechanism is called the weighted VCG mechanisms. Hence, another way to state our result is that if $\mathbb{D}$ is open and connected and f is neutral then the only strategy-proof mechanisms are the weighted VCG mechanisms.

AUCTION DOMAINS ARE NOT OPEN. It is well known that in auction domains there are social choice functions other than affine maximizers which are implementable (Lavi et al., 2003). These social choice functions are also neutral. Hence, in auction domains, there are neutral social choice functions which are implementable but not weighted welfare maximizers. This is reconciled from the fact that auction domains are restricted domains which are not necessarily open. For example, consider the sale of two objects to two buyers. The set of allocations can be $\{a, b, c, d\}$, where a denotes buyer 1 gets both the objects, b denotes buyer 2 gets both the objects, c denotes buyer 1 gets object 1 and buyer 2 gets object 2, and d denotes buyer 1 gets object 2 and buyer 2 gets object 1. Note here that in every utility vector t^a for allocation a buyer 2 will have zero valuation. Similarly, in every utility vector t^b for allocation b buyer 1 will have zero valuation. Hence, this domain is not open.

A DISCONNECTED BUT OPEN $\mathbb{D}$. Consider an example with two agents and three alternatives a, b, and c. Figure 1 shows $\mathbb{D}$. Figure 1 shows possible types of both agents on every alternative. As shown in Figure 1, $\mathbb{D} = S_1 \cup S_2$, where S_1 and S_2 are two open squares in $\mathbb{R}^2$.

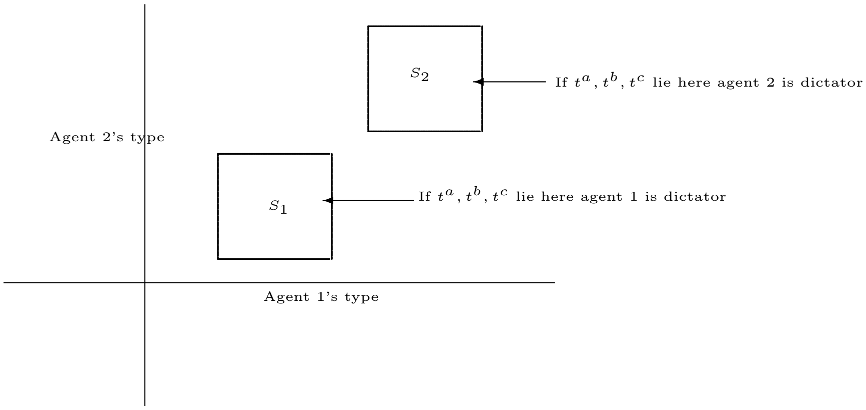

Figure 1: A disconnected but open $\mathbb{D}$

Now, consider the following social choice function f with corresponding payment rule. For any $t = (t^a, t^b, t^c)$,

1) if t^a, t^b, t^c lie in S_1, then agent 1 is the dictator (zero payments for both agents),

2) if t^a, t^b, t^c lie in S_2, then agent 2 is the dictator (zero payments for both agents),

3) else, we chose efficiently, i.e., $f(t) \in \arg\max\{t_1^a+t_2^a, t_1^b+t_2^b, t_1^c+t_2^c\}$ (any VCG payment).

This social choice function is clearly implementable but not an affine maximizer.

No ordering without neutrality. If we drop neutrality and replace it with non-imposition, then Roberts' theorem says that affine maximizers (as in Theorem 1) are the only implementable social choice functions. But affine maximizers do not necessarily induce the ordering we discussed. This is because of the $\kappa(\cdot)$ terms in the affine maximizers. For example, consider a type profile $t = (t^a = x, t^b = y, t^{-ab})$. Suppose $a \in C^f(t)$ but $b \notin C^f(t)$. Here, the $\kappa(a)$ term may be higher than $\kappa(b)$ such that when we permute the columns of a and b and get the new type profile $s = (s^a = y, s^b = x, t^{-ab})$, we still have $a \in C^f(s)$ and $b \notin C^f(s)$. Thus, our social welfare ordering is not induced here.

Anonymity gives efficiency. Consider the following additional condition on every social choice function.

Definition 9 *A social choice function f is* **anonymous** *if for every $t \in \mathbb{T}$ and every permutation σ on the row vectors (agents) of t we have $f(\sigma(t)) = f(t)$.*

Definition 10 *An ordering R on $\mathbb{D}$ satisfies* **anonymity** *if for every $x, y \in \mathbb{D}$ and every permutation σ on agents we have xIy if $x = \sigma(y)$.*

Lemma 7 *Suppose f is implementable and anonymous. Then, R^f satisfies anonymity.*

Proof: Consider $x, y \in \mathbb{D}$ such that $y = \sigma(x)$ for some permutation σ on agents. Assume for contradiction xP^fy. Consider a type profile t such that $t^a = x$ and $t^b = y$ for all $b \neq a$. Hence, $C^f(t) = \{a\} = f(t)$. Since f is anonymous $f(\sigma(t)) = a$. Hence, $\sigma(x) = yR^f\sigma(y)$. Repeating this argument again, we will get $\sigma(y)R^f\sigma(\sigma(y))$. Clearly, after applying the permutation σ a finite k number of times on y, we will get the x vector again. This will lead to xP^fx, which is a contradiction. ■

It is straightforward to show using Theorem 3 that if $\mathbb{D}$ is open and connected than every implementable, neutral, and anonymous social choice function is the efficient social choice function. Here, we show that this result holds for some other domains too. The beautiful proof of this fact is an adaptation of a proof by Milnor Milnor (1954) (see also Theorem 4.4 in d'Aspremont and Gevers (2002)), and is given in the appendix.

Theorem 4 *Suppose f is implementable, neutral, and anonymous. If $\mathbb{D} = [0, H)$, where $H \in \mathbb{R} \cup \{\infty\}$, then f is the efficient social choice function.*

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Appendix

Proof of Fact 1

Proof: Let $x, y \in X$ and $z = \alpha x + (1-\alpha)y$ for some $\alpha \in (0,1)$. We consider two possible cases.

Case 1: Suppose X is closed in $\mathbb{D}$. Assume for contradiction that $z \notin X$. Since D is convex, $z \in \mathbb{D} \setminus X$. Since X is closed in $\mathbb{D}$, the set $\mathbb{D} \setminus X$ is open in $\mathbb{D}$. Hence, $\mathbb{D} \setminus X$ is open in $\mathbb{R}^n$. This means, there exists an n-dimensional open ball $B_\delta(z) = \{z' : \|z' - z\| < \delta\}$ of radius δ such that every $z' \in B_\delta(z)$ belongs to $\mathbb{D} \setminus X$.

Now, consider an iterative procedure as follows. Let l, h be two variables in $\mathbb{R}^n$. Initially, set $l = x$ and $h = y$. In every step,

- if z is in the convex hull of l and $\frac{l+h}{2}$ then set $h = \frac{l+h}{2}$,
- else set $l = \frac{l+h}{2}$.

If $\|l - h\| < 2\delta$, stop. Else, repeat the step.

Since $\|l - h\|$ strictly decreases in every step, the procedure will terminate. Moreover, l and h at the end of the procedures are two points in X. Hence, $\frac{l+h}{2}$ is in X and lies in the ball $B_\delta(z)$. This is a contradiction [2].

Case 2: Suppose X is open in $\mathbb{D}$. Then X is open in $\mathbb{R}^n$. This implies that there exists an open ball $B_{\delta_x}(x)$ around x of radius δ_x and an open ball $B_{\delta_y}(y)$ around y of radius δ_y such that each of these balls are contained in X. Let $\delta = \min(\delta_x, \delta_y)$. Using the fact that for every $x' \in B_{\delta_x}(x)$ and every $y' \in B_{\delta_y}(y)$ we have $\frac{x'+y'}{2} \in X$, we get that every $x'' \in B_\delta(\frac{x+y}{2})$ lies in X. Now, we can repeat the procedure of Case 1 to conclude that $z \in X$. ■

Proof of Theorem 4

Proof: Note that $\mathbb{D}$ is open from above and a meet-semilattice. Hence, by Proposition 1, R^f is an ordering. By Proposition 2 and Lemma 7, f satisfies weak Pareto, invariance, and anonymity (we do not need continuity for this proof). Also, note that for any $x \in \mathbb{D}$, any permutation of the elements of x results in a vector in $\mathbb{D}$.

Now, choose $x, y \in \mathbb{D}$ such that $\sum_{i \in N} x_i = \sum_{i \in N} y_i$. By anonymity, we can rearrange x and y in non-decreasing order but mutually ranked the same way as x and y. Considering successively, in these new vectors, each pair of corresponding components and subtracting from each the minimal one, we get again two new vectors which are ranked the same way as x and y by invariance (note here that these two new vectors belong to $\mathbb{D} = [0, H)^n$). Repeating these two operations at most n times, we will reach two zero vectors (since $\sum_{i \in N} x_i = \sum_{i \in N} y_i$). Hence, $xI^f y$.

Next, we show that if $\sum_{i \in N} x_i > \sum_{i \in N} y_i$ then $xP^f y$. Let $\delta = \frac{1}{n}[\sum_{i \in N} x_i - \sum_{i \in N} y_i]$. Consider the vector z defined as $z_i = y_i + \delta$ for all $i \in N$. By weak Pareto $zP^f y$. Further $\sum_{i \in N} x_i = \sum_{i \in N} z_i$. Hence, $xI^f z$. Hence, $xP^f y$.

By Lemma 3, for every $t \in \mathbb{T}$, we have $f(t) \in C^f(t)$. Hence, $f(t) R^f a$ for all $a \in A$. Hence, f is the efficient social choice function. ■

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[2] Essentially, the procedure generates a sequence of dyadic rational numbers. We know that the set of dyadic rational numbers are dense. Since X is closed, we are done.