Measurement of Group Differential
- A formal treatment for two-group case

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Abstract

This paper proposes a set of measures to explain group-differential. Mishra and Subramanian (2006) developed axioms of DBLS and RBLS for group-differential measures which is sensitive to levels in the sense that a given hiatus at lower levels of failure (or higher levels of attainment) is considered worse off. Mishra (2007) refines these axioms and adds the axiom of normalization. This paper critically reviews these axioms and develops a comprehensive axiomatic characterization of group-differential. Also, the paper evaluates the strengths and weaknesses of the group-differential measures developed in literature, and in the process proposes a set of new measures for both attainment and failure. Empirical illustration of group differential measures with MDG goal (Goal 4: *Infant and Child Mortality*) *related two indicators; % children immunized (as an example of attainment indicators) and infant mortality rate (as a failure indicators) for different regions of the world is also provided.*

1. Introduction

Group differential is an important class of measures to know the difference in failure[1] (or attainment) between two groups. This differential can be expressed by simple difference or simple ratio or can be complicated further. Whether simple or complex, the measure needs to satisfy certain properties. Mishra and Subramanian (2006) have introduced two axioms on level sensitivity, difference-based level sensitivity (DBLS) and and ratio-based level sensitivity (RBLS). These axioms indicate that for a failure (attainment) indicator a given hiatus between two groups should acquire a greater salience the higher (lower) the level at which the hiatus arises. They discuss three existing and a fourth new measure of group differential, which were later refined and enriched by Mishra (2007), who also added the axiom of normalization.

This paper will deviate from the past literatures in two respects. First, this will treat attainment and failure differently. This is essential as a group differential measure based on a failure indicator satisfying certain axiom of level sensitivity may fail for the same axiom if indicator is converted to attainment by taking inverse. Same is true for attainment indicator also. So for group differential measure, unlike as perceived in the literature (Mishra (2007), Mishra and Subramanian (2006)) converting a failure indicator to attainment applying additive inverse and vice versa should be avoided. Secondly, this paper unlike the past ones, develop over an attainment indicator (the corresponding analysis of failure indicators are left to appendices). This is done as failure indicators are

[1] There are genuine failure indicators like Infant Mortality Rate (IMR), Maternal Mortality Rate (MMR), and Death Rate. There are genuine attainment indicators like literacy rate and income. An attainment indicator can be converted as failure by taking its inverse, like when literacy rate is replaced with illiteracy rate, or in case of income, a maximum may be posited and the actual observations subtracted from this to obtain an indicator of failure.

difficult to comprehend than attainment. Failure indicators are like negative numbers, where higher (lower) the absolute value, worse (better) the situation. For attainment, understanding becomes easy as higher means improvement, and lower means decline.

In this paper, some new axioms are introduced. The measures that already developed in the literature are revisited. Also new measures are brought in both in failure and attainment point of view. Empirical illustration has been provided with % of children immunized and infant mortality rate for different regions of the world as indicators of failure and attainment respectively.

2. Axiomatic characterization of group differential

Consider a socio economic attainment indicator[2], $I_{js}\varepsilon[0,1]$; 0=no attainment and 1=full attainment for j^{th} group ($j=a,b$), under situation s ($s=A,B$). Without loss of generality, given a situation s let group a be considered to be at higher attainment level than b, $I_{as}>I_{bs}$ and given a group j situation A is at least as good as B so that $I_{jA}\geq I_{jB}$. Following are a number of intuitive properties that a measure of group differential, d or $d(I_{as},I_{bs})$ should satisfy.

Normalization (Axiom N): At a basic level, the measure of group differential should lie between zero and unity, $d\varepsilon[0,1]$. At a fundamental level the measure should have a minimum and a maximum such that 0=no group-differential and 1=highest group-differential. Highest group-differential occurs when one of the groups is at minimum level of attainment and the other is at maximum or vice versa.

[2] The axioms are outlined for failure indicator I' in Appendix 1.

Strong Monotonicity (Axiom M): The measure of group differential should be such that it is higher (lower) if one of the groups remaining constant at a particular level of attainment; the other changes so that the absolute gap increases (decreases). Mathematically, $d(I_{aA},I_{bA})>d(I_{aB},I_{bB})$ when $I_{aA}=I_{aB}$ and $I_{bA}<I_{bB}$. Weak monotonicity means, $d(I_{aA},I_{bA})\geq d(I_{aB},I_{bB})$ when $I_{aA}=I_{aB}$ and $I_{bA}<I_{bB}$. Two corollaries of strong monotonicity are axioms of minimality and maximality.

Minimality (Axiom M_{min}): The measure of group differential should be higher than its minimum value if there is some group differential. Mathematically, $d>0$ if $(I_{as}-I_{bs})>0$.

Maximality (Axiom M_{max}): The measure of group differential should be lower than its maximum value if the group-differential is less than the highest. Mathematically, $d<1$ if $(I_{as}-I_{bs})<1$.

Difference based level sensitivity (DBLS) (Axiom D): The measure of group differential should be such that it is more pronounced if the difference level persists at a higher level of attainment. Mathematically, if $I_{aA}-I_{bA}\geq I_{aB}-I_{bB}=h$; $h>0$, then the DBLS axiom requires that $d(I_{aA},I_{bA})>d(I_{aB},I_{bB})$.

Ratio based level sensitivity (RBLS) (Axiom R): The measure of group differential should be such that it is more pronounced if the ratio level persists at a higher level of attainment. Mathematically, if $I_{aA}/I_{bA}\geq I_{aB}/I_{bB}=k$; $k>0$, then the RBLS axiom requires that $d(I_{aA},I_{bA})>d(I_{aB},I_{bB})$.

For attainment, DBLS is the stricter condition than RBLS, as when $I_{aA}-I_{bA}=I_{aB}-I_{bB}$, $I_{aA}/I_{bA}<I_{aB}/I_{bB}$. On the contrary, for failure indicators, RBLS is stricter.

3. Measures of group differential

There are four differential measures indicated in Mishra and Subramanian (2006) and the measures are reconstructed in Mishra (2007) and an extension of the fourth measure is proposed. The five measures dealt in the literature are the following,

$$d_1=I_a-I_b \quad (1)$$

$$d_2=I_a^{\delta}-I_b^{\delta};\ 0<\delta<1 \quad (2)$$

$$d_3=1-I_b/I_a \quad (3)$$

$$d_4=1-I_b^{\alpha+1}/I_a^{\alpha};\ \alpha>0 \quad (4)$$

$$d_5=1-I_b^{\alpha+\delta}/I_a^{\alpha};\ \alpha>0,\ \delta\geq 0 \quad (5)$$

All the above measures are developed for failure indicators in mind. But keeping in mind that RBLS is a stricter condition for failure indicators, d_1 and d_2 would not be considered as serious contenders of a differential measure for failure. In similar lines, the ratio based indicators like d_3, d_4 and d_5 are not intuitive for differential measure for attainment indicators. So, let us have a fresh look on the group differential measures and organize measures for attainment and for failure.

3.1 Measure of group differential for attainment indicators

3.1.1 Simple difference [I_a-I_b]

Simple difference is a powerful differential measure for its simplicity. It satisfies axiom of normalization both at basic and fundamental level, axiom of strong monotonicity and its corollaries of maximality and minimality. It satisfies RBLS and weakly satisfies DBLS. This has been discussed by Mishra and Subramaniyam (2006), but in the context of failure indicators; hence found to violate the axioms of DBLS and RBLS. It is a poor differential measure of failure, but it is a powerful measure for attainment satisfying all axioms except DBLS, which it satisfies weakly.

3.1.2 Measure based on simple difference and exponent [I_a^{α}-I_b^{β}; 0<α≤β]

The simple difference can be improvised using exponent to make it satisfy DBLS strongly. This is achieved by raising the lower value to a power at least as high as the higher value. This is a general class of measure where d_1 and d_2 turn out to be special cases (d_2 results when α=β=δ; and d_1, when α=β=1). This measure, except under the special condition of α=β=1, satisfies all the axioms. However, this class of measure is dependent on the subjective choice of α and β and the actual value of measure can vary significantly with small change in the values of α and β.

3.1.3 Alternative measure [$d_6=(I_a-I_b)/(1-I_b/k)$; $k>1$]

Here the numerator is same as d_1 which satisfies DBLS weakly and all other the axioms. ($1-I_b/k$) is divided to increase the hiatus at higher level of attainment so that *DBLS* would be satisfied in a strong sense. The rationale for choosing I_b over I_a in the denominator expression is I_b being the lower value, a division by inverse of the value will cause minimum deformation to d_1. For $k<1$, normalization will be violated. For $k=1$, strong monotonicity and maximality corollary are violated at $I_a=1$ as $I_a\rightarrow 1$, $d_6\rightarrow 1$ irrespective the value of I_b. Also, when $I_b,I_a\rightarrow 1$, $d_6\rightarrow 0/0$, which is undefined. So $k>1$ is the preferred option. Higher the value k, lower is the deformation to d_1. As $k\rightarrow\infty$, $d_6\rightarrow d_1$ and as $k\rightarrow 1$, $d_6\rightarrow(I_a-I_b)/(1-I_b)$ i.e. maximum deformation possible. For simplicity of expression lets us assume $k=2$ to define d_7 which is a special case of d_6.

$$d_7=(I_a-I_b)/(1-I_b/2)$$

3.2 Measure of group differential for failure indicators

3.2.1 Simple ratio $[1-I_b/I_a]$

Likewise simple difference, simple ratio is a powerful differential measure for its simplicity. In fact, Mishra and Subramnayam (2006) initially proposed the indicator to be I_b/I_a, which was later modified by Mishra (2007) to $1-I_b/I_a$ to make it satisfy normalization. The form proposed by Mishra (2007) is more appropriate as it satisfies strong monotonicity except when $I_b=0$. It satisfies DBLS and weakly satisfies RBLS. It is a powerful differential measure of failure satisfying all axioms except DBLS, which it satisfies weakly.

3.2.2 Measure based on simple ratio and exponent $[1-I_b^{\beta}/I_a^{\alpha};\ 0<\alpha\leq\beta]$

The simple ratio can be improvised using exponent to make it satisfy RBLS strongly. This is achieved by raising the lower value to a power at least as high as the higher value. This is a general class of measure where d_3, d_4 and d_5 turn out to be special cases (d_4 results when $\beta=\alpha+1$; d_5 when $\beta=\alpha+\delta$; and d_3, when $\alpha=\beta=1$). This class of measure satisfies DBLS, RBLS except under the special condition of $\alpha=\beta=1$, and strong monotonicity except when $I_b=0$ (when $I_b\rightarrow 0$, $d_3\rightarrow 1$ irrespective the value of I_a). However, this class of measure fails in normalization at fundamental level when $I_a=I_b$. For example, at $I_a=I_b$, $d_4=1-I_b$ and $d_5=1-I_b^{\delta}$, so both $d_4,d_5<1$. Also, like the case of attainment, this class of measure, is dependent on the subjectivity of α and β.

3.2.3 Alternative measures $[d_8=(1-I_b/I_a)(1-I_b/k);\ k\geq 1]$

Likewise d_6, the basis of d_8 is simple ratio i.e. d_3, which satisfies RBLS weakly and all other the axioms. $(1-I_b/k)$ is multiplied to increase the hiatus at lower level of failure so that

RBLS would be satisfied in a strong sense. The logic of choosing I_b over I_a is same as in the case of d_6. However, unlike d_6, here $k \geq 1$. At $k=1$, there is maximum deformation to d_3. Higher the value k, lower is the deformation to d_1. This class of measure is an improvement over d_4, d_5 as it satisfies normalization and over d_3 as it satisfies RBLS strongly. The simplest expression with $k=1$ be defined d_9 which is a special case of d_8.

$$d_9=(1-I_b/I_a)(1-I_b)$$

A point to note here, all the differential measures for failure fail in strong monotonicity and maximality corollary at $I_b=0$. In fact, at $I_b=0$ strong monotonicity and RBLS conditions are contradictory to each other, as the former says with increase in I_a the differential measure should increase, whereas RBLS says the reverse. However, like earlier measure as well as in the proposed case, at $I_b=0$, the measure becomes unity irrespective of value of I_a.

3.3 Summary table

Table 1 summarizes the axioms applied to all group-differential measures discussed in the previous sections. Both in the case of attainment and failure, there is an improvement in the axiomatic properties of measures.

Table 1: Axioms applied to all group differential measure

Group differential Measures for Attainment Indicators	**Axioms**					
	N	**M**	**M_{min}**	**M_{max}**	**D**	**R**
$d_1 = I_a - I_b$	Yes	Yes	Yes	Yes	Yes (weakly)	Yes
$d_2 = I_a^{\delta} - I_b^{\delta}$	Yes	Yes	Yes	Yes	Yes	Yes
$d_7 = (I_a - I_b)/(1 - I_b/2)$	Yes	Yes	Yes	Yes	Yes	Yes
Group differential Measures for Failure Indicators	**Axioms**					
	N	**M**	**M_{min}**	**M_{max}**	**D**	**R**
$d_3 = 1 - I_b/I_a$	Yes	Yes ($I_b \neq 0$)	Yes	Yes ($I_b \neq 0$)	Yes	Yes* (weakly)
$d_4 = 1 - I_b^{\alpha+1}/I_a^{\alpha}$; $\alpha > 0$	Yes (Basic)#	Yes ($I_b \neq 0$)	Yes	Yes ($I_b \neq 0$)	Yes	Yes*
$d_5 = 1 - I_b^{\alpha+\delta}/I_a^{\alpha}$; $\alpha > 0$, $\delta \geq 0$	Yes (Basic)#	Yes ($I_b \neq 0$)	Yes	Yes ($I_b \neq 0$)	Yes	Yes*
$d_9 = (1 - I_b/I_a)(1 - I_b)$	Yes	Yes ($I_b \neq 0$)	Yes	Yes ($I_b \neq 0$)	Yes	Yes*

satisfies normalization at a basic level, d_4, d_5 ε[0,1], but fails it at a fundamental level because at no differential, $0 < I_a = I_b < 1$, $d_4, d_5 > 0$.
* At $I_b = 0$, RBLS gets satisfied in a weak sense

4. Empirical Illustration

For empirical illustration, indicators related to Million Development Goal (MDG) related to infant and child mortality is used (WB, 2008). Two indicators are chosen, one of attainment kind i.e. % children immunized, and one failure i.e. infant mortality rate. The data for different regions of the world is provided for 1990 and 2005. The % of children immunized is expressed in fraction and the IMR data is computed by dividing the value by 1000 so that I_a and I_b gets scaled to the 0-1 range.

Table 2: Infant mortality rate and immunization rate for different regions of the world (1990, 2005)

Regions	Infant mortality rate (in fraction)		Immunization rate (in fraction)	
	1990	2005	1990	2005
World	**0.064**	**0.051**	**0.73**	**0.77**
High income	0.009	0.006	0.83	0.93
Low & middle income	0.069	0.056	0.72	0.75
East Asia & Pacific	0.043	0.026	0.90	0.83
Europe & Central Asia	0.039	0.027	0.84	0.96
Latin America & Carib.	0.043	0.026	0.76	0.92
Middle East & N. Africa	0.060	0.043	0.83	0.92
South Asia	0.086	0.062	0.56	0.64
Sub-Saharan Africa	0.109	0.096	0.57	0.64

From above data, some relevant cases are picked to demonstrate the advantage of proposed measures over the earlier ones. For attainment indicator i.e. immunization rate, Table 3 gives the group differential under various measures. The case chosen is where the difference between the two groups is small. As expected, d_1, which is simple difference, does not able to capture the level sensitivity. d_2 though captures the same, can vary significantly with the value of δ. The proposed measure d_7 values the group difference in 1990 as 0.375, which has increased to 0.412 in 2005, though the absolute difference of child immunization rate has remained same.

Table 3: Comparing group differential under various measures using Child Immunization rate for different regions of the world

Cases & Groups information	Situations	I_a	I_b	d_1	d_2	d_7
Case 1 $I_{aA} - I_{bA} \approx I_{aB} - I_{bB}$ *a*: Middle East & N. Africa *b*: South Asia	*B*: 1990	0.830	0.560	0.270	0.159^{*} $0.035^{\#}$ $0.258^{\$}$	0.375
	A: 2005	0.920	0.640	0.280	0.163^{*} $0.038^{\#}$ $0.252^{\$}$	0.412

Notes: d_1, d_2, and d_7 are the differential measures for attainment indicators discussed in the text. d_2 has been computed for different δ values where *, #, $ corresponds 0.5, 0.1, 0.9 respectively.

Table 4 compares the group differential measures for failure indicator i.e. infant mortality rate for different regions. Case 1 represents two situations with ratio I_b/I_a almost same. Expectedly, d_3, the measure based on simple ratio does not capture the level sensitivity, where as all other measures do. It has been shown how d_4 and d_5 can vary considerably with the choice of different values of α, δ. Case 2 demonstrates how d_4 and d_5 fail in normalization.

Table 4: Comparing group differential under various measures using Child Immunization rate for different regions of the world

Cases	Groups	Situations	I_a	I_b	d_3	d_4	d_5	d_9
Case 1 $I_{bA}/I_{aA} \approx I_{bB}/I_{aB}$ *a*: South Asia *b*: Middle East & N. Africa		*B*: 1990	0.086	0.060	0.302	0.421^{*} $0.499^{\#}$ $0.650^{\$}$	0.206^{*} $0.337^{\#}$	0.279
		A: 2005	0.062	0.043	0.304	0.585^{*} $0.642^{\#}$ $0.752^{\$}$	0.235^{*} $0.363^{\#}$	0.396
Case 2 $I_{aA}=I_{bA}<I_{aB}=I_{bB}$ *a*: East Asia & Pacific *b*: Latin America & Carib.		*B*: 1990	0.043	0.043	0	0.570	0.081	0
		A: 2005	0.026	0.026	0	0.740	0.126	0

Notes: d_3, d_4, d_5, and d_9 are the differential measures for failure indicators discussed in the text. d_4 has been computed for different α values where *, #, $ represents 0.1, 0.5, 1.5 respectively. d_5 has been computed for different (α,δ) values where *,# represents (0.5,0.01) and (1.0,0.1) respectively.

5. Conclusions

This paper discusses on the group differential indicators. It recognizes the limitation of the past work and suggest the need for a different treatment for finding group differential in attainment and failure indicators. In the axiomatic characterization of group differential measure, axiom of 'strong monotonicity' with its corollaries of 'minimality' and 'maximality' was added to the existing axioms of level sensitivities and normalization. Simple difference turns out to be a better measure for group differential in attainment indicators whereas simple ratio is more suitable for failure indicators. New indicators are proposed both for group differential in attainment and group differential in failure. The advantages of the proposed indicators are presented in axiomatic characterization. Unlike simple difference and simple ratio, the proposed indicators strongly satisfy the level sensitivities. Also, the new indicators do not suffer for the limitation faced by the past indicators caused by subjectivity of the choice of exponents. The advantages of the proposed indicators are empirically demonstrated with MDG goal indicators % children immunized (for attainment), and infant mortality rate (for failure).

Appendix 1

Consider a socio economic failure indicator, $I_{js}\varepsilon[0,1]$; 0=no failure and 1=complete failure for j^{th} group (j=a,b), under situation s (s=A,B). Without loss of generality, given a situation s let group b be considered to be at lower failure level than a, $I_{as}>I_{bs}$ and given a group j situation A is at least as good as B so that $I_{jA}\leq I_{jB}$. Following are a number of intuitive properties briefly explained that a measure of group differential, d or $d(I_{as},I_{bs})$ should satisfy.

Normalization (Axiom N): At a basic level, $d\varepsilon[0,1]$. At a fundamental level, 0=no group-differential and 1=highest group-differential.

Strong Monotonicity (Axiom M): $d(I_{aA},I_{bA})>d(I_{aB},I_{bB})$ when $I_{aA}=I_{aB}$ and $I_{bA}<I_{bB}$. Weak monotonicity means, $d(I_{aA},I_{bA})\geq d(I_{aB},I_{bB})$ when $I_{aA}=I_{aB}$ and $I_{bA}<I_{bB}$.

Minimality (Axiom M_{min}): $d>0$ if $(I_{as}-I_{bs})>0$.

Maximality (Axiom M_{max}): $d<1$ if $(I_{as}-I_{bs})<1$.

Difference based level sensitivity (DBLS) (Axiom D): If $I_{aA}-I_{bA}\geq I_{aB}-I_{bB}=h$; $h>0$, then $d(I_{aA},I_{bA})>d(I_{aB},I_{bB})$.

Ratio based level sensitivity (RBLS) (Axiom R): If $I_{aA}/I_{bA}\geq I_{aB}/I_{bB}=k$; $k>0$, then $d(I_{aA},I_{bA})>d(I_{aB},I_{bB})$.

References

Mishra, U. S. and S. Subramanian (2006) On Measuring Group-differentials Displayed by Socio-economic Indicators, *Applied Economics Letters*, 13, 519-521.

Mishra S. (2007) On Measuring Group-differentials Displayed by Socio-economic Indicators, *Applied Economics Letters*, 13, 519-521.