The Games 'Oil'igopolists Play

Neil D'Souza *
National University of Singapore.

Aditya Goenka †
National University of Singapore.

September 2009

Abstract

The theory of 'oil'igoply developed by Loury (1986) predicts that countries holding larger proved reserves of oil tend to produce quantities that are larger in absolute size but smaller as a proportion of their reserves. However, the underpinnings of the theory rest on oil producers following open loop strategies and the assumption of the existence of a downward sloping demand curve for oil. We address these issues via a Cournot-Walras and a fully strategic model that are essentially two-period, deterministic, simultaneous-move dynamic games in the spirit of Cordella and Dutta (2002). The equilibria in both models are subgame perfect and both capture the empirical regularity predicted by the theory of 'oil'igopoly. The Cournot-Walras model is, in addition, able to capture regularities in the data not alluded to by the theory. The fully strategic model dispenses with the need to assume a downward sloping demand curve and hence provides an alternative approach to the partial equilibrium framework in which the problem has been studied. The fully strategic model can also generate positive correlation between the price and quantity of oil traded, which is found in the data.

Keywords: Dynamic games; exhaustible resources; lattice programming; submodular games, supermodular games.

JEL Classification: C62, C73, D51, Q31, Q41.

*Department of Economics, National University of Singapore, AS2, Level 6, 1 Arts Link, Singapore 117570, Email: g0600645@nus.edu.sg

†Department of Economics, National University of Singapore, AS2, Level 6, 1 Arts Link, Singapore 117570, Email: g0600645@nus.edu.sg

1 Introduction

The theory of 'oil'igopoly developed by Loury (1986) and extended by Polasky (1992) gives a simple yet elegant prediction, that countries holding larger proved reserves of oil tend to produce quantities that are larger in absolute size but smaller as a proportion of their reserves. Both Polasky (1992) and Pickering (2008) have found support for this reserves-production relationship when using data for oil-producing countries and companies. The theory thus provides valuable insight into the resource extraction problem whilst being consistent with a seemingly robust empirical regularity. However, we believe that there are some technical and conceptual aspects that need to be addressed, which would deepen our understanding of the resource extraction problem.

The technical aspects include the use of open loop strategies and the partial equilibrium framework. The theory was derived using the Nash equilibrium, open loop concept rather than the subgame perfect equilibrium concept. In general, an open loop strategy is time-dependent and is independent of the state of the world at which an agent chooses her action. Open loop strategies, which in the absence of binding contracts, lend themselves to characterizing equilibria that are not subgame perfect. Thus the use of Nash equilibrium to dynamic games gives rise to dynamic inconsistency and substantial differences in the time path of production as noted by Reinganum and Stokey (1985). Partial equilibrium models of imperfect competition generally rely on convenient assumptions regarding the shape and slope of the inverse demand curves. The assumptions typically lend themselves to ensuring the existence of equilibria or the uniqueness of Nash equilibrium. The advantage of such assumptions is the analytic tractability that they afford. However, their theoretical underpinnings are suspect (Grandmont, 1993). For the resource extraction problem under consideration, we do run into an identification problem when estimating the demand function for oil. Its resolution would require identifying restrictions that would necessitate alterations to the model used to study the problem. This would bring about added complexity when using the partial equilibrium framework.

On the conceptual side, there are two issues. First, the reserves-production relationship predicted by the theory leads to the conclusion that the quantity of oil produced by an oil-producing country depends on its own reserves. Econometric analysis for individual countries shows that while this might be true for a few countries, in general, a country's production decisions also depend on the reserves and the production decisions of other producing countries. Thus, the reserves-production relationship possesses a richer structure than that predicted by the theory of 'oil'igopoly. Second, oil is used by most producing countries as a means of obtaining consumption goods. Increases in the price of goods and services consumed by the producing nations would most certainly have an impact on their extraction decisions. This aspect of trade affecting the decisions of the oil producers has never been studied in a model that reflects the empirical regularity of the reserves-production relationship posited by the theory of 'oil'igoply.

In this paper we study two models. They are both deterministic, simultaneous-move dynamic games in the spirit of Cordella and Datta (2002). The models differ in whether we assume that there exists a downward sloping inverse demand curve for oil or not. We call the model where we assume a downward sloping inverse demand curve to exist as the *Cournot-Walras* model. The model where we dispense with this assumption and explicitly introduce trade is referred to as the *fully strategic* model. In both the models, there is trade between oil producers and the traded consumption good (foodgrain) producers. The oil producers have a preference for foodgrains and the foodgrain producers have a preference for oil. Hence the necessity for trade in the model.

The oil producers are to assumed to behave strategically in both the models. In the Cournot-Walras model, the producers of foodgrains are deemed to be competitively supplying foodgrains whereas in the fully strategic model the foodgrains producers also behave strategically. To highlight the differences between the two approaches, we focus our attention on the two-period model. In the Cournot-Walras model, the oil producers trade each period in the spot market. There is no futures market. Each oil producer has a finite amount of resource endowment that she must deplete over two periods. A producer's problem concerns how much of her endowment to trade while taking into account the production decisions of her fellow producers. The idea of strategic substitutability provides the intuition for the behavior observed in the Cournot-Walras model. In addition to being able to explain the reserves-production relationship, the Cournot-Walras characterization of the behavior of oil-producing countries shows that the production decisions of an oil producer depend not only on her reserves but also the production decisions of the other producers and their reserves.

In the fully strategic model, the story is similar, except for the presence of strategic foodgrain producers as well. Here, the oil producers and the foodgrain producers trade each period in the spot market. Each oil producer has a finite amount of resource endowment that she must deplete over two periods. The same goes for the foodgrain producers. The problem for each agent concerns how much of their endowment to trade given that their production decision affects the production decisions of the other producers, be they oil or foodgrain producers. Here, both strategic substitutability and complementarity help us intuit the outcome of the model. In addition to being able explain the reserves-extraction relationship posited by the theory of 'oil'igopoly, we are able to capture the possibility of a coordination failure between oil producers and the foodgrain producers. This might offer some insight into the existence of oil price shocks.

Furthermore, we are able to capture both the *market* externality, wherein a producer's production decision affects relative prices, as well as the *dynamic* externality, wherein a producer's decision affects the future amount of resource stock. We obtain subgame perfect equilibria by construction. We use lattice programming techniques to prove the existence of equilibrium, the presence of unicity of the equilibrium or its multiplicity, the stability of the equilibrium as well as do some comparative static exercises. We find that strategic substitutability and complementarity of the production decisions of the producers and nature of the products, the limited supply of crude oil and implicit constraint on the foodgrain producers, drive the results of the model.

To summarize, we reconsider the reserves-production relationship predicted by the theory of 'oil'igopoly. The relationship predicted by the theory is robust; however, the data offers some more regularities that need to be explained. The Cournot-Walras model is consistent with the predicted reserves-production relationship and also provides an explanation of the other regularities. However, the Cournot-Walras model relies on the assumption of a downward sloping demand curve. It is not clear why this should hold. Therefore, we consider the fully strategic model that does not require this assumption. We find that the fully strategic model provides a richer framework than the Cournot-Walras model to understand the resource extraction problem.

The literature on oil production considers that understanding the strategic interactions between the producers of the resource is the key to understanding the price and output path of oil production. We find that this is only partially true. Oil being a global commodity whose price fluctuations have large repercussions on the economy only weakly fits into the partial equilibrium framework in which it is studied. There seems to be a larger 'game' going on, that between the oil producers and the non-oil producers. When we do consider this interaction, we are still able to match a robust empirical regularity of the resource extraction problem. Furthermore, to a certain extent, both the models considered here provide justification for the belief that OPEC is a loose cartel, whose members' production decisions seem to be directed more by their endowments rather than some sinister need to hold the world to ransom.

In section 2 we test the robustness of the reserves-output relationship and determine whether the data holds some other regularities. Section 3 describes the Cournot-Walras model. We provide a description of the model, an analysis of the model, and an example. We also provide some proofs regarding the existence of equilibria, unicity or multiplicity of equilibria, and some comparative static results. We empirically analyze the oil price-output relationship to ascertain whether the assumption of the downward sloping demand curve holds true in section 4. Section 5 presents a description of the fully strategic model. We provide a description of the model, an analysis of the model, and an example. Section 6 consists of the conclusion.

2 Empirical analysis

The 'oil'igopoly model of Loury (1986) predicts that oil producers endowed with large resource stocks produce a larger amount but a smaller percentage of their stocks than producers endowed with small resource stocks. We first estimate the relationship posited by the theory. We then determine whether the data throws light on any other regularities. For instance, one might assume that the output of an oil producing nation is not just tied to its own reserves. It might be reasonable to posit that the reserves of other producing nations as well as their contemporaneous output decisions might affect a nation's production decisions.

The data used are from British Petroleum Statistical Review of World Energy (2008).[1] These consist of proven reserves as of the end of year, and average daily extraction rates [2] in thousands of millions of barrels (tmb) for every major oil extracting country in the world annually over the period 1980–2007. Not all the countries included have been operational through the entire time period and the full data set consists of 1026 country–year observations for extraction and remaining reserves, covering 38 countries, 11 of which are members of OPEC. [3] The reserves data correspond to reserves contemporaneously known about and economically viable given prevailing market conditions, and is distinct from the idea of a fixed resource stock. A caveat though, the reserves data are difficult to collect and subject to measurement error.

To utilize the full data set and employ panel data estimation the econometric specification is proposed:

[1] Available online at http://www.bp.com/centres/energy

[2] Multiplied by 365 to give a figure for annual extraction rates.

[3] Algeria, Indonesia, Iran, Iraq, Kuwait, Libya, Nigeria, Qatar, Saudi Arabia, Venezuela and the United Arab Emirates.

$$\frac{O_{it}}{o_{it}} = \alpha_i + \beta_i O_{it} + u_{it}$$

(where O_{it} represents the reserves of country i in time period t and o_{it} represents the oil produced by country i in time period t)

Estimation was performed using panel-data techniques and the results reported in Table 1 pertain to a common intercept term, fixed-effects, or random-effects estimation.

Table 1: Panel-data estimation

	Common intercept	Fixed Effects	Random Effects
β	7.00E-07	5.59E-07	6.38E-07
t-Statistic	16.03362	3.768143	6.267141
R^2	0.199118	0.462109	0.442955

We see that in all the cases estimated, the value of β is positive and significant. This is line with the panel estimates obtained by Polasky (1992) and Pickering (2008). The relationship between the reserves production ratio and the reserves is indeed robust.

It is reasonable to assume that the decisions of an oil-producing country rest not only on its reserves but also the reserves of other oil-producing nations and their contemporaneous production decisions. We could consider this to be an extension of the theory of 'oil'igopoly. Consider the the following equations:

$$\frac{O_t^i}{o_t^i} = \alpha + \beta O_t^i + \gamma O_t^{-i} + \delta o_t^{-i} + u_t$$

(where O_t^i represents country $i's$ reserves in time period t, o_t^i represents country $i's$ oil production in time period t, O_t^{-i} represents the cumulative reserves of producers other than country i in time period t, and o_t^{-i} represents the cumulative amount of oil produced by producers other than country i in time period t)

Table 2 presents an interesting picture. The extension of the theory to include the effects of the production decisions of the other producers and their reserves improves the estimation though not by much. However, the variables O_t^{-i} and o_t^{-i} are significant at the 5% level of significance. We do find that it is not just the reserves of a nation that affect its reserves to production ratio.

Table 2: Panel-data estimation: 'Oil'igopoly and an extension

Coefficient	'Oil'igopoly	t-Statistic	Extension	t-Statistic
α	20.57206	8.952487	19.96705	1.430587
β	7.00E-07	16.03362	6.69E-07	14.77172
γ	-	-	-6.84E-08	-2.610279
δ	-	-	3.09E-06	2.250020
adj. R^2	0.198344		0.202059	

The reserves of the other nations and their production decisions matter. This empirical finding does give us a clue as to direction that our model ought to take, that of a dynamic game between oil producing nations and this is what we consider in the next section.

3 The Cournot-Walras model

3.1 A description

Consider an economy with two goods, one being oil and the other being foodgrains. The foodgrains sector is competitive whilst the oil sector is not. The foodgrain sector thus consists of Walrasian producers whereas the oil sector is characterized by Cournot producers, who take into consideration the impact of their output decisions on the relative price of foodgrains and oil. There is heterogeneity amongst the oil producers who produce a homogenous good. The heterogeneity comes from the differences in their initial endowments. Let there be $n > 1$ such oil producers. Here, n is a positive integer. In each period, the oil producer has to decide the amount of oil to extract such that she can enjoy consumption of foodgrains via trade in the spot market. There is no futures market for either of the goods. Furthermore, the model is purely deterministic.

Let the amount of oil that oil producer $i \in \{1, \ldots, n\}$ has when she begins time period $t \in \{1, 2\}$ be O_t^i. The amount of stock extracted by the producer i in period t is o_t^i, where $o_t^i \in [0, O_t^i]$. Since we assume the market for food to be competitive, the oil producers take as given the price of food p_t in each period. The period utility function is given by $U : \Re_+ \rightarrow \Re$. We assume U to be a twice differentiable, continuous function. The intertemporal utility function for each producer i is given by $V^i = \Sigma_{t=1}^{2} \delta^{t-1} U(f_{tC}^i)$, where f_{tC}^i symbolizes the consumption of foodgrain by agent i in period t. The individual budget constraint faced by a producer i in period t is $q_t o_t^i = p_t f_{tC}^i$ for all $i \in \{1, \ldots, n\}$ and $t \in \{1, 2\}$. We consider that the inverse demand curve for oil to be downward sloping and concave, and dependent on the production decisions of all the n oil producers. We have $q_t : \Re_+^n \rightarrow \Re_+$.

The agent i maximizes the discounted sum of utilities subject to the budget constraint and technical feasibility of her choice. The agent's intertemporal optimization problem is thus

$$max_{f_{1C}^i} \Sigma_{t=1}^{2} \delta^{t-1} U(f_{tC}^i) \tag{1}$$

subject to

$$q_t o_t^i = q_t f_{tC}^i$$
$$0 \leq o_t^i \leq O_t^i$$
$$o_1^i + o_2^i \leq O_1^i$$

(where the first constraint is the individual budget constraint, the second constraint is the intratemporal resource constraint, and the third constraint is the intertemporal resource constraint)

We have in effect described an n-player dynamic game that is completely described by the tuple $(n, \theta, g, U, \delta)$, where n is the set of players, θ is the set of states, that is, the levels of stock of the resource of each agent and the price schedule of food in the two periods, U is the instantaneous utility function common to all the players and δ is the discount factor that is common to all the players. The state space space $[0, O_i^t]$ of each agent $i \in \{1, \ldots, n\}$ is a compact interval in R_+. The transition function $g : \Re_+ \rightarrow \Re_+$ represents the depletion of the resource stock over time. The assumptions on g are that it is a linear function in two arguments (the resource stock and the extracted amount in the previous period), it is decreasing in the amount extracted in the previous period and it is increasing in the resource stock in the previous period. The objective of each player is to maximize the discounted sum of utilities while taking the action choices of the other players as given. The choices of the other players enter into a player's utility maximization problem via the inverse demand function. The problem that we consider differs from the dynamic resource games studied by Levhari and Mirman (1980), Sundaram (1989) and others as unlike their case of a common resource, here the state variable is multidimensional (individual stocks of the resource).

An intertemporal Cournot-Walras equilibrium is a Nash equilibrium of the above-mentioned game. Thus we have,

Definition 1 *An intertemporal Cournot-Walras equilibrium is the n-tuple of strategy decisions $(o_1^{i*})_{i=1}^n$ for the n agents, such that for each player i we have*

$$V^i(o_1^{i*}, (o_1^{k*})_{k=1,k\neq i}^n) \geq V^i(o_1^i, (o_1^{k*})_{k=1,k\neq i}^n)$$

$$\textit{(for all } 0 \leq o_t^i \leq O_t^i\textit{)}$$

The maximization problem can be converted into a one-shot problem because every agent is going to consume all of her stock in the final period. This allows us to fold the game back via backward induction and solve the resulting problem. Also, since we employ a discrete-period, finite-horizon game, the subgame perfect Nash equilibria is obtained via backward induction. For the sake of completeness, a brief account of the lattice programming techniques employed, a few theorems regarding existence and uniqueness of equilibrium and an account of the manner in which our model would fit into the framework is described in the appendix.

3.2 The dynamic resource extraction problem

Submodularity implies strategic substitutability and is captured succinctly as the cross-partial of the value function (with respect to the action of the other player) being negative. This implies that the extra value of playing a higher action is lower when the other player plays a higher action. The resource extraction problem we have considered fits into the framework of submodularity. This is evident when we consider the value function.

$$V^i = V_1^i + V_2^i$$

This is the sum of the value function for the agent i in the two periods.

Proposition 2 *The value function is submodular in the action space of the agents if it can be represented as the product of a submodular function and an increasing function or the product of a submodular function and a decreasing function.*

Proof. Submodularity of V^i implies that both the period value functions V_1^i and V_2^i are submodular. The period value functions depend on the choice variables (which are the decisions of how much to extract in each period) and the parameters. We write the period value functions as $V_1^i(x, y; \theta)$ and $V_2^i(x, y; \theta)$, where x and y represent the choice variables of agent i and the cumulative sum of the actions of the other agents respectively and θ is the vector of parameters. Suppressing the parameter symbol and the time and agent subscript and superscript, we represent the value functions as functions of the choice variables as

$$V(x, y) = l(x, y)h(x)$$

Suppose that $x^{'} > x$ and $y^{'} > y$. For $V(x, y)$ to be submodular, it would imply from the definition of submodularity that

$$V(x^{'}, y^{'}) - V(x, y^{'}) \leq V(x^{'}, y) - V(x, y)$$

$$g(x^{'}, y^{'})l(x^{'}) - g(x, y^{'})l(x) \leq g(x^{'}, y)l(x^{'}) - g(x, y)l(x)$$

$$[g(x^{'}, y^{'}) - g(x^{'}, y)]l(x^{'}) \leq [g(x, y^{'}) - g(x, y)]l(x) \tag{2}$$

Thus, for equation (2) to hold, either:

- $g(\cdot, \cdot)$ is submodular in (x, y) and $l(\cdot)$ is increasing in x that is, $[g(x^{'}, y^{'}) - g(x^{'}, y)] \leq [g(x, y^{'}) - g(x, y)]$ and $l(x^{'}) \leq l(x)$ or
- $g(\cdot, \cdot)$ is submodular in (x, y) and $l(\cdot)$ is decreasing in x that is, $[g(x^{'}, y^{'}) - g(x^{'}, y)] \leq [g(x, y^{'}) - g(x, y)]$ and $l(x^{'}) \geq l(x)$

■

Thus, the value function for an agent being submodular is dependent on it being representable as the product of a submodular function (in her choice variable and that of the other agents) and an increasing or decreasing function (in her own choice variable). It turns out that the deterministic transition function and the form of the inverse demand function make the value function submodular.

We can generalize the result. Consider the value function as follows:

$$V^i = max_{f^i_{1C}, f^i_{2C}}[U_1(f^i_{1C}) + \delta U_2(f^i_{2C})]$$

Using the individual budget constraints we obtain

$$V^i = max_{o^i_1, o^i_2} U_1(\frac{q_1 o^i_1}{p_1}) + \delta U_2(\frac{q_2 o^i_2}{p_2})$$

Thus, we can write

$$V^i = max_{o^i_1, o^i_2} U_1(o^i_1, \sum_{j \neq i} o^j_1; p_1) + \delta U_2(o^i_2, \sum_{j \neq i} o^j_2; p_2) \quad (3)$$

The period utility functions in equation (3) will be submodular if $\frac{\partial U_1}{\partial o^i_1 \partial \sum_{j \neq i} o^j_1} \leq 0$ and $\frac{\partial U_2}{\partial o^i_2 \partial \sum_{j \neq i} o^j_2} \leq 0$

This is possible if the utility is an increasing function in the choice variables and the product of the choice variable and the inverse demand function is submodular. This happens due to the deterministic transition function that is standard in the literature. In fact, the standard Cournot oligopoly model exhibits the property of submodularity as described above.

To illustrate the 'oil'igopoly result, let us assume that the inverse demand curve is downward sloping and linear, the assumption that is standard in the literature. This implies that $q_1 = A - o^i_1 - o^{-i}_1$ and $q_2 = A - O^i_1 - O^{-i}_1 + o^i_1 + o^{-i}_1$ (where, $A > O^i_1 + O^{-i}_1$, $o^{-i}_1 = \sum^n_{j=1, j \neq i} o^j_1$, and $O^{-i}_1 = \sum^n_{j=1, j \neq i} O^j_1$) . Then, differentiating equation (3) above we get,

$$\begin{aligned} o^i_1 &= \frac{(\frac{U'_1(\cdot)}{p_1} - \frac{\delta U'_2(\cdot)}{p_2})A}{(\frac{U'_1(\cdot)}{p_1} + \frac{\delta U'_2(\cdot)}{p_2})2m} \\ &+ \frac{(\frac{2\delta m U'_2(\cdot)}{p_2})}{(\frac{U_1(\cdot)}{p_1} + \frac{\delta U_2(\cdot)}{p_2})2m} O^i_1 \\ &+ \frac{(\frac{\delta n U'_2(\cdot)}{p_2})}{(\frac{U_1(\cdot)}{p_1} + \frac{\delta U_2(\cdot)}{p_2})2m} O^{-i}_1 \\ &- \frac{(\frac{U_1(\cdot)}{p_1} + \frac{\delta U_2(\cdot)}{p_2})n}{(\frac{U_1(\cdot)}{p_1} + \frac{\delta U_2(\cdot)}{p_2})2m} o^{-i}_1 \end{aligned}$$

The equation is of the form $o_1^i = a + bO_1^i + cO_1^{-i} - do_1^{-i}$ (where, b, c, and, d are positive quantities as we consider an increasing utility function). From this we can get $\frac{\partial \frac{O_1^i}{o_1^i}}{\partial o_1^j} > 0$, $\frac{\partial \frac{O_1^i}{o_1^i}}{\partial O_1^j} < 0$, and $\frac{\partial \frac{O_1^i}{o_1^i}}{\partial O_1^i} > 0$. The signs match those in Table 2. We are thus able to extend the partial equilibrium model of 'oil'igopoly such that it captures some other empirical relationships that have a bearing on the reserves-production relationship.

Now, given that value function is submodular, we are able to show via Theorem 9 that the best response correspondence is non-empty and that submodularity, which implies decreasing differences, gives downward sloping reaction functions. Then, we are able to show that there exists a Nash equilibrium of the game in pure strategies via Theorem 3. Theorem 4 gives us the conditions that should hold for the equilibrium to be unique. Theorem 5 concerns the structure of the equilibrium set in the case of a unique equilibrium.

Theorem 3 *There exists a Nash equilibrium of the game in pure strategies.*

Proof. We impose the condition that the agent has to exhaust her resource stock in the final period. With this condition and given the deterministic transition function, we can fold the game such that we obtain a one-shot game. As the value function of the one shot game for the agent i is submodular in the product of the strategy spaces of the agents and the strategy sets and the parameter sets obey the requirements of Theorem 10, then we can say that the best response correspondence is a non-empty, compact poset of $\Re_+$ and admits a greatest element. Furthermore, we obtain that the best response correspondence is decreasing in the cumulative actions of the other agents. Now the set of players is finite and the utility of each player is continuous in the actions of the agent and that of the sum of the other agents, i.e., we are considering an aggregative game. Hence, using the fixed-point theorem for decreasing mappings from Kukushkin (1994) there exists a Nash equilibrium of the game in pure strategies. ■

Theorem 4 *The equilibrium is unique if the period value functions can be represented as the product of a concave, downward sloping inverse demand curve and a concave, upward sloping function of the agent's output decision or the product of a concave, upward sloping inverse demand curve and a concave, downward sloping output function of the agent's output decision*

Proof. Since we are considering an aggregative game, we can use a corollary arising from Corchón's uniqueness theorem (Folmer and von Mouche, 2004) to give us the restrictions essential for uniqueness. Let y stand for sum of the actions of the agents at the Nash equilibrium and let x be the output of player i at the Nash equilibrium. Let the value function be represented by

$$V^i = p_1(y)f_1(x) + p_2(y)f_2(x)$$

We have to determine the properties of the functions $p(\cdot)$ and $q(\cdot)$ that will give us a unique equilibrium. This occurs when

$$t^i(x,y) = \frac{\partial V^i(x,y)}{\partial x} + \frac{\partial V^i(x,y)}{\partial y}$$

is strictly decreasing in x and decreasing in y.

Differentiating t^i with respect to y we get

$$\frac{\partial t^i(x,y)}{\partial y} = p_1^{''}(y)f_1(x) + p_1^{'}(y)f_1^{'}(x) + p_2^{''}(y)f_2(x) + p_2^{'}(y)f_2^{'}(x) \tag{4}$$

Differentiating t^i with respect to x we get

$$\frac{\partial t^i(x,y)}{\partial x} = p_1^{'}(y)f_1^{'}(x) + p_1(y)f_1^{''}(x) + p_2^{'}(y)f_2^{'}(x) + p_2(y)f_2^{''}(x) \tag{5}$$

Thus, for equations (4) and (5) to be decreasing and strictly decreasing respectively, we require that $p_1(y)$ is concave and downward sloping, $p_2(y)$ is concave, upward sloping, $f_1(x)$ is concave, upward sloping, and $f_2(x)$ is concave and downward sloping.

■

Theorem 5 *Consider the value function to be nonincreasing in the endogenous variables and weakly increasing in exogenous parameters. Then, the equilibrium set in a non-empty, complete lattice, if and only if, there is a unique equilibrium.*

Proof. See Roy and Sabarwal (2007). For our scenario, in the case of a unique equilibrium, this is trivially satisfied. ■

3.3 An example

Let us, for the sake of simplicity consider that there are two types of oil producers, those that have a large initial endowment of oil (agents i) and those that have a small initial endowment of oil (agents j). Let the ones that have a larger endowment number m and the others number n. Let us assume a linear inverse demand function $q_t = A - mo_t^i - no_t^j$. The period budget constraint is given by $q_t o_t^i = p_t c_{t,f}^i$. Then, we can write down the value function of a player of type i as

$$V^i = max_{c_{1,f}^i, c_{2,f}^i}[(c_{1,f}^i)^\alpha + \delta(c_{2,f}^i)^\alpha]$$

Using the budget constraints we obtain

$$V^i = max_{o_1^i, o_2^i}\frac{(q_1 o_1^i)^\alpha}{(p_1)^\alpha} + \frac{(q_2 o_2^i)^\alpha}{(p_2)^\alpha}$$

We impose the condition that the agent has to exhaust her resource in the final period, that is, $o_2^i = O_1^i - o_1^i$ in conjunction with the assumed functional form of the price function, $q_t = A - mo_t^i - no_t^j$, where $A > mO_t^i - nO_t^j$. The value function then becomes,

$$V^i = max_{o_1^i}((A - mo_1^i - no_1^j)(o_1^i)/(p_1))^\alpha + ((A - mo_2^i - no_2^j)(o_2^i)/(p_2))^\alpha \quad (6)$$

Differentiating equation (6) with respect to o_1^i and o_1^j we find that

$$\begin{aligned}\frac{\partial V^i}{\partial o_1^i} &= \\ &-\frac{\alpha m(A - mo_1^i - no_1^j)^{\alpha-1}(o_1^i)^\alpha}{(p_1)^\alpha} \\ &+\frac{\alpha(A - mo_1^i - no_1^j)^\alpha(o_1^i)^{\alpha-1}}{(p_1)^\alpha} \\ &+\frac{\delta\alpha m(A - mO_1^i - nO_1^j + mo_1^i + no_1^j)^{\alpha-1}(O_1^i - o_1^i)^\alpha}{(p_2)^\alpha} \\ &-\frac{\delta\alpha(A - mO_1^i - nO_1^j + mo_1^i + no_1^j)^\alpha(O_1^i - o_1^i)^{1-\alpha}}{(p_2)^\alpha}\end{aligned} \quad (7)$$

and

$$\frac{\partial V^j}{\partial o_1^j} =$$
$$- \frac{\alpha n(A - mo_1^i - no_1^j)^{\alpha-1}(o_1^j)^{\alpha}}{(p_1)^{\alpha}}$$
$$+ \frac{\alpha(A - mo_1^i - no_1^j)^{\alpha}(o_1^j)^{\alpha-1}}{(p_1)^{\alpha}}$$
$$+ \frac{\delta\alpha n(A - mO_1^i - nO_1^j + mo_1^i + no_1^j)^{\alpha-1}(O_1^j - o_1^j)^{\alpha}}{(p_2)^{\alpha}}$$
$$- \frac{\delta\alpha(A - mO_1^i - nO_1^j + mo_1^i + no_1^j)^{\alpha}(O_1^j - o_1^j)^{1-\alpha}}{(p_2)^{\alpha}} \qquad (8)$$

Setting equations (7) and (8) to zero, we can plot the reaction functions. Differentiating equation (6) twice with respect to o_1^i we find that

$$\frac{\partial^2 V^i}{(\partial o_1^i)^2} =$$
$$\frac{\alpha(\alpha-1)m^2(A - mo_1^i - no_1^j)^{\alpha-2}(o_1^i)^{\alpha}}{(p_1)^{\alpha}}$$
$$- \frac{\alpha^2 m(A - mo_1^i - no_1^j)^{\alpha-1}(o_1^i)^{\alpha-1}}{(p_1)^{\alpha}}$$
$$- \frac{\alpha^2(A - mo_1^i - no_1^j)^{\alpha-1}(o_1^i)^{\alpha-1}}{(p_1)^{\alpha}}$$
$$+ \frac{\alpha(\alpha-1)(A - mo_1^i - no_1^j)^{\alpha}(o_1^i)^{\alpha-1}}{(p_1)^{\alpha}}$$
$$+ \frac{\delta\alpha(\alpha-1)m^2(A - mO_1^i - nO_1^j + mo_1^i + no_1^j)^{\alpha-2}(O_1^i - o_1^i)^{\alpha}}{(p_2)^{\alpha}}$$
$$- \frac{\delta\alpha^2 m(A - mO_1^i - nO_1^j + mo_1^i + no_1^j)^{\alpha-1}(O_1^i - o_1^i)^{\alpha-1}}{(p_2)^{\alpha}}$$
$$- \frac{\delta\alpha^2 m(A - mO_1^i - nO_1^j + mo_1^i + no_1^j)^{\alpha-1}(O_1^i - o_1^i)^{\alpha-1}}{(p_2)^{\alpha}}$$
$$+ \frac{\delta\alpha(\alpha-1)m(A - mO_1^i - nO_1^j + mo_1^i + no_1^j)^{\alpha}(O_1^i - o_1^i)^{1-\alpha}}{(p_2)^{\alpha-2}} \qquad (9)$$

From the above equation we note that

$$\frac{\partial^2 V^i}{(\partial o_1^i)^2} \leq 0$$

This implies that the agent i is maximizing her value function with regard to her choice variable o_1^i. Furthermore,

$$\begin{aligned}
\frac{\partial^2 V^i}{\partial o_1^i \partial o_1^j} = & \\
& \frac{\alpha(\alpha-1)mn(A - mo_1^i - no_1^j)^{\alpha-2}(o_1^i)^{\alpha-1}}{(p_1)^\alpha} \\
& - \frac{\alpha^2 n(A - mo_1^i - no_1^j)^{\alpha-1}(o_1^i)^{\alpha-1}}{(p_1)^\alpha} \\
& + \frac{\delta\alpha(\alpha-1)mn(A - mO_1^i - nO_1^j + mo_1^i no_1^j)^{\alpha-2}(O_1^i - o_1^i)^\alpha}{(p_2)^\alpha} \\
& - \frac{\delta\alpha^2(A - mO_1^i - nO_1^j + mo_1^i no_1^j)^{\alpha-1}(O_1^i - o_1^i)^{\alpha-1}}{(p_2)^\alpha}
\end{aligned} \tag{10}$$

From the above equation we note that

$$\frac{\partial V^i}{\partial o_1^i \partial o_1^j} \leq 0$$

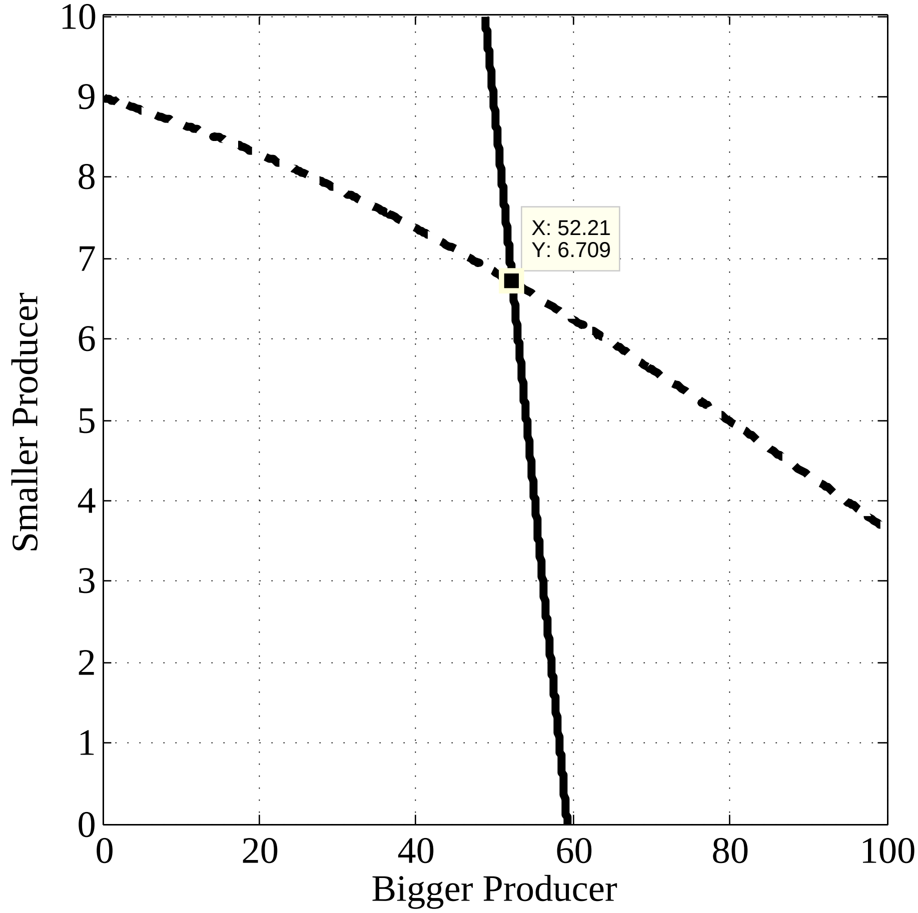
This implies strategic substitutability in the production decisions of the two types of agents. Equations (9) and (10) define the second order and cross-partial derivatives of the value function respectively.

We consider the following parameter values to plot figure 1:

$$\begin{array}{lll}
\alpha = 0.7 & \delta = 0.8 & A = 2000 \\
p_1 = 1 & p_2 = 1 & m = 10 \\
n = 30 & O_1^i = 100 & O_1^j = 10
\end{array}$$

The reaction functions can be seen in Figure 1 below. The solid curve gives the reaction function of the agents of type i and the dashed curve gives the reaction function of the agents of type j. The number of agents of each type and the reserve values are roughly chosen to reflect the lopsided endowments of crude oil observed in the real world. This translates into approximately 75 percent of the world's reserves lying in the hands of OPEC. The two reaction functions are downward sloping and intersect at a point. Here we see that the prediction of theory of 'oil'igopoly hold. The agents of type j, the ones with the small endowment, produce approximately 67 percent of their endowment in the first period. The agents of type i, the ones with the large endowment, produce a little over a half of their endowment in the first period. The downward sloping reaction functions are a result of the strategic substitutability of the actions of the two types of agents. As we can see, it is this combined with the endowments that drives the result.

Figure 1



10
9
8
7
6
5
4
3
2
1
0
X: 52.21
Y: 6.709
Smaller Producer
0
20
40
60
80
100
Bigger Producer

4 The downward sloping inverse demand curve

The Cournot-Walras model matches the stylized fact posited by the theory of 'oil'igopoly. However, the Cournot-Walras model relies on the assumption of the existence of a downward sloping inverse demand curve to achieve this. Here we seek to establish whether or not this assumption is valid.

We obtain data from the British Petroleum Statistical Review of World Energy (2008).[4] Two data series are considered: the quantity of oil consumed on a yearly basis during 1965-2007 and the price of oil during 1965-2007 in 2007 US Dollars. We denote the quantity consumed in period t by q_t and the price of oil in period t by p_t. We first check whether the series are stationary. We employ the augmented Dickey-Fuller test on p and q. The results are shown in Table 3.

Table 3: Test for integration

Data Series	ADF Test Statistic
p	-0.161302 {1}
q	-2.032745 {1}

None of the reported t-statistics for p and q for all variants of the test equation (with an intercept term, a trend and intercept term, or lagged variables) are significant at even the 1% level of significance, implying that the null-hypothesis of a unit root cannot be rejected for these variables. We consider the cointegrating equation between p and q. Since both the data series are integrated of order 1, we can obtain cointegration between the two series. This would rule out the possibility of obtaining a spurious regression. We use the Johansen cointegration test with a linear deterministic trend to obtain the cointegrating regression. The trace test and the maximum-eigen value test both indicate the presence one cointegrating regression at the 5% level of significance (this result holds for other formulations of the test). This means that there exists a long-run stationary relationship between the variables. The results are mentioned below in Table 4.

Contrary to the downward sloping inverse demand curve that is considered in the literature, the inverse demand curve seems to be upward sloping as coefficient of q is positive. The data does not back the assumption of the downward

[4]Available online at http://www.bp.com/centres/energy

Table 4: Cointegration equation

	p	q	trend
Normalized cointegrating coefficients	1	120.9170	-8.631761
Standard errors	-	(35.0519)	(3.22600)

sloping demand curve. We essentially face an identification problem. The data points that we observe are equilibrium points and the upward sloping demand curve necessitates the assumption that both the demand and the supply curve are shifting. It is not clear whether these assumptions are justifiable. Furthermore, the shifts in the curves cannot be captured without the use of identifying restrictions, which are not a part of the framework considered when studying the theory of 'oil'igopoly. Hence, an alternative would be to study the problem in a framework where one circumvents the problem of having to assume a downward sloping demand curve. We do this in the fully strategic model.

5 The fully strategic model

5.1 A description

As before we consider an economy in which there are two types of goods, foodgrains and oil. It is a two-period model. In this model, both the producers of foodgrains as well as the producers of oil act strategically. It is in this sense that we consider this model to be a fully strategic model.

Consider the producers of oil. They receive an endowment of oil in the first period. They gain utility each period by consuming foodgrains. They procure foodgrains by trade each period where they trade oil for foodgrain. They have to decide how much oil to trade in the first and second periods. The production technology of the oil producers is just a storage technology where oil that is not extracted is stored in the ground for future use. We thus have heterogeneity amongst the oil producers, the heterogeneity coming from their initial endowments. The model is purely deterministic.

An oil producer $i \in \{1, \ldots, n\}$ has O_t^i of oil at the beginning of time period $t \in \{1, 2\}$. The amount of stock extracted for trade by the producer i in period t is represented by o_{tT}^i, where $o_{tT}^i \in [0, O_t^i]$. The intertemporal utility function for each producer i is given by $V^i = \Sigma_{t=1}^2 \delta^{t-1} U(f_{tC}^i)$, where f_{tC}^i symbolizes the consumption of food by agent i in period t. The individual budget constraint faced by the producer in each period is $q_t o_{tT}^i = p_t f_{tC}^i$ for all $i \in \{1, \ldots, n\}$ and $t \in \{1, 2\}$. Here, the total value of the good exchanged in each period determines the relative price of oil and foodgrains.

The agent i maximizes the discounted sum of utilities subject to the budget constraint and technical feasibility of her choice. The agent's intertemporal optimization problem is thus

$$max_{f_{tC}^i} \Sigma_{t=1}^2 \delta^{t-1} U(f_{tC}^i) \tag{11}$$

subject to

$$q_t o_{tT}^i = q_t f_{tC}^i$$

$$0 \leq o_t^i \leq O_t^i$$

$$o_1^i + o_2^i \leq O_1^i$$

and

$$p_t \Sigma_j (f_t^j - f_{tP}^j) = q_t \Sigma_i (o_t^i - o_{tP}^i)$$

(where the first constraint is the individual budget constraint, the second represents the intratemporal resource constraint, the third represents the intertemporal resource constraint, and the fourth constraint gives the market clearing condition)

In this game, the feasible strategy of an agent is the choice of the resource extracted in the first period, which in turn is dependent on her resource stock, the stock of the other agents, and the discount factor δ, such that $0 \leq o_t^i \leq O_t^i$, $o_1^i + o_2^i \leq O_1^i$ and $q_t o_{tT}^i = p_t f_{tC}^i$. Agents choose their strategies simultaneously and the payoffs for the oil producing agents are given by

$$V^i(o_1^i, (o_1^k)_{k=1,k\neq i}^n, (f_1^j)_{j=1}^m) = max_{f_{1C}^i, f_{2C}^i}[U(f_{1C}^i) + \delta U(f_{2C}^i)]$$

It must be noted that the agent $i's$ consumption of food in every period is given by

$$f_{tC}^i = \frac{q_t o_t^i}{q_t}$$

from the budget constraints, for all $i \in \{1, \ldots, n\}$. Additionally, agents take into account the influence of their extraction decisions on the relative price of oil and food. An intertemporal Cournot equilibrium is a Nash equilibrium of the above-mentioned game. Thus we have,

Definition 6 *An intertemporal Cournot equilibrium for the producer of oil is the n-tuple of strategy decisions $(o_1^{i*})_{i=1}^n$ for the n agents, such that for each player i we have*

$$V^i(o_1^{i*}, (o_1^{k*})_{k=1,k\neq i}^n, (f_1^{j*})_{j=1}^m) \geq V^i(o_1^i, (o_1^{k*})_{k=1,k\neq i}^n, (f_1^{j*})_{j=1}^m)$$

(for all $0 \leq o_t^i \leq O_t^i$)

Consider the producers of foodgrains. They gain utility each period by consuming oil, which they trade for foodgrains each period. They receive an endowment of foodgrain in the first period. They have to decide how much oil to trade in each period to procure oil for consumption. The foodgrain that is not traded in the current period is used to produce foodgrain for the next period. We thus have heterogeneity amongst the foodgrain producers, the heterogeneity coming from their initial endowments. The model is purely deterministic.

Let the quantity of foodgrain that a producer producer $j \in \{1, \ldots, m\}$ has at time period $t \in \{1, 2\}$ be represented by f_t^j. The amount of foodgrain traded by the producer j in period t is represented by f_{tT}^i, where $f_{tT}^i \in [0, f_t^j]$. The rest is used in production for the next period and is represented by f^i_{tP}. Thus, $f^i_{t+1} = w(f^i_{tP})$, where $w : \Re_+ \rightarrow \Re_+$ and w satisfies the usual regularity conditions. In every period we have $f^i_{tP} + f^i_{tT} = f^j_t$. The intertemporal utility function for each producer j is given by $V^j = \Sigma^2_{t=1}\delta^{t-1}U(o^j_{t,C})$, where $o^j_{t,C}$ symbolizes the consumption of oil by agent j in period t. The budget constraint faced by the producer in each period is then $q_t o^j_{t,C} = p_t f^j_{t,T}$ for all $j \in \{1, \ldots, m\}$ and $t \in \{1, 2\}$. Here, the total value of the good exchanged in each period determines the relative price of oil and foodgrains.

The agent j maximizes the discounted sum of utilities subject to the budget constraint and technical feasibility of her choice. The agent's intertemporal optimization problem is thus

$$max_{o^j_{1C}, o^j_{2C}} \Sigma^2_{t=1}\delta^{t-1}U(o^j_{tC}) \tag{12}$$

subject to

$$q_t o^j_{tC} = q_t f^j_{tT}$$
$$0 \leq f^j_{tT} \leq f^j_t$$
$$f^j_{tT} + f^j_{tC} = f^j_t$$

and

$$p_t\Sigma_j(f^j_t - f^j_{tP}) = q_t\Sigma_i(o^i_t - o^i_{tP})$$

(where the first constraint represents the individual budget constraint, the second is the intratemporal resource constraint, the third is the intertemporal resource constraint, and the fourth constraint represents the market clearing condition)

For a foodgrain producer, the feasible strategy consists of the choice of the food to be traded in the first period, f^j_{1T}, which in turn is dependent on her endowment in the first period f^j_1, the endowments of the food and oil producers, the price schedule of food in the two periods q_t for $t \in \{1, 2\}$ and the discount factor δ. Agents choose their strategies simultaneously and the payoffs for the foodgrain producing agents are given by

$$V^j(f^j_{1T}, (f^k_{1T})^m_{k=1,k\neq j}, (o^i_1)^n_{i=1}) = max_{o^j_{1C}, o^j_{2C}} \Sigma^2_{t=1}\delta^{t-1}U(o^j_{tC})$$

It must be noted that the agent $j's$ consumption of oil in every period is given by

$$o^j_{tC} = \frac{p_t f^j_{tT}}{q_t}$$

from the budget constraints, for all $j \in \{1, \ldots, m\}$. Additionally, agents take into account the influence of their extraction decisions on the relative price of oil and food. An intertemporal Cournot equilibrium is a Nash equilibrium of the above-mentioned game. Thus we have,

Definition 7 *An intertemporal Cournot equilibrium for the foodgrain producer is the m-tuple of strategy decisions $(f_{1T}^{j*})_{j=1}^{m}$ for the m agents, such that for each player j we have*

$$V^j(f_{1T}^{j*}, (f_{1T}^{k*})_{k=1,k\neq i}^{m}, (o_1^{i*})_{i=1}^{n}) \geq V^j(f_{1T}^{j}, (f_{1T}^{k*})_{k=1,k\neq i}^{m}, (o_1^{i*})_{i=1}^{n})$$

$$\textit{(for all } 0 \leq f_{tT}^{j} \leq f_t^j \textit{)}$$

The maximization problem can be converted into a one-shot problem because each agent is going to consume all of their stock in the final period. This allows us to fold the game back via backward induction and solve the resulting problem.

Given our description of the game above we find that one level, the producers of the two goods, foodgrains and oil, are are engaged in a game where the complementarities of their production decisions result in the possibility of multiple equilibria from this purely deterministic model. They are thus engaged in a supermodular game. At another level, oil producers and foodgrain producers are engaged in a strategic game with players of their own type. The game being played between players of the same type could take on the nature of a supermodular or submodular game depending on the parameter values. In the literature thus far, it is only the interactions between the oil producers that has been modeled. The trade approach to the oil market with fully strategic agents seems to provide a richer description of behavior of the oil producing nations and the environment.

To illustrate the 'oil'igopoly result, let us substitute the constraints given in equation (11) into the utility function and differentiating with respect to o_1^i we get,

$$o_1^i \quad = \quad \frac{\frac{U_1^{'}(\cdot)\sum_j(f_1^j - f_{1P}^j)}{(\sum_i(O_1^i - O_{1P}^i))} + \frac{U_2^{'}(\cdot)\sum_j(f_{1P}^j)}{(\sum_i(O_{1P}^i))}}{\frac{U_1^{'}(\cdot)\sum_j(f_1^j - f_{1P}^j)}{(\sum_i(O_1^i - O_{1P}^i))^2} + \frac{U_2^{'}(\cdot)\sum_j(f_{1P}^j)}{(\sum_i(O_{1P}^i))^2}} + \frac{\frac{U_1^{'}(\cdot)\sum_j(f_1^j - f_{1P}^j)}{(\sum_i(O_1^i - O_{1P}^i))^2}}{\frac{U_1^{'}\sum_j(f_1^j - f_{1P}^j)}{(\sum_i(O_1^i - O_{1P}^i))^2} + \frac{U_2^{'}\sum_j(f_{1P}^j)}{(\sum_i(O_{1P}^i))^2}})O_1^i$$

The equation is of the form $o_{1P}^i = a + bO_1^i$. From this we can obtain $\frac{\partial \frac{O_1^i}{o_{1P}^i}}{\partial O_1^i} > 0$, which matches the empirical regularity posited by the theory of 'oil'igopoly.

The theorems 8 to 10 mentioned in the appendix still hold when we consider a submodular game. For proofs of existence, uniqueness and stability in a supermodular game, we can appeal to the theorems in Chapter 2 of Vives (1999). We find that for the the game between the foodgrain producers and the oil producers, there exists a Nash equilibrium in pure strategies. The production decisions of the foodgrain producers and the oil producers are strategic complements. Thus, we cannot rule out the possibility of multiple equilibria. Given that we observe strategic complementarity of actions, the reaction functions are upward sloping.

For the game between the oil producers, we can use the results that we have mentioned in Section 3.1. The production decisions of the oil producers are strategic substitutes. The reaction curves are downward sloping and under certain conditions we can obtain a unique equilibria. The model is deterministic, we still obtain the possibility of coordination failure amongst the agents. In the example that follows, we see that for the interactions between the oil and foodgrain producers we have three equilibrium points, one of which is a stable equilibrium point.

5.2 An example

5.2.1 The foodgrain producer's problem

Let the producers of food be labeled by j and the producers of oil be labeled by i. Let the number of grain producing agents be m and those producing oil be n, where $m, n > 1$. Let f_t^j denote the stock of grain in period t. Let f_{tP}^j denote the stock of grain used in production in period t and f_{tT}^j the amount of foodgrains used to trade for oil. Since each agent of type j is endowed with an initial stock of grain, f_1^j is given for each agent of type j. We put no restrictions on the initial endowment for the agents except that they are positive. The endowment that is put aside for production is used to produce the good next period. Thus we write:

$$f_t^j = (f_{t-1P}^j)^\beta \tag{13}$$

(where $\beta < 1$)

Also, it must be that

$$f_t^j = f_{tP}^j + f_{tT}^j \tag{14}$$

The agent's budget constraint is given by

$$q_t o^j_{tC} = p_t f^j_t \tag{15}$$

(where, p_t is the price of grain, q_t is the price of oil and o^j_{tC} is the amount of oil consumed by the agent of type j in period t).

It must be that the value of the goods traded in each period must be equal. Thus we have

$$p_t \Sigma_f (f^j_t - f^j_{tP}) = q_t \Sigma_i (o^i_t - o^i_{tP}) \tag{16}$$

Each agent thus solves the following maximization problem,

$$V^j = max_{o^j_{1C}, o^j_{2C}} \Sigma^2_{t=1} \delta^{t-1} (o^j_{tC})^\alpha \tag{17}$$

From equations (15) and (17) we get,

$$V^j = max_{f^j_{1T}, f^j_{2T}} \Sigma^2_{t=1} \delta^{t-1} (\frac{p_t}{q_t} f^j_{tT}))^\alpha$$

$$V^j = max_{f^j_{1T}, f^j_{2T}} (\frac{p_1}{q_1} f^j_{1T})^\alpha + \delta(\frac{p_2}{q_2} f^j_{2T})^\alpha \tag{18}$$

From equations (16) and (18) we get

$$\begin{aligned} V^j = & max_{f^j_{1P}} \frac{(\Sigma^n_{i=1}(o^i_1 - o^i_{1P}))^\alpha}{(\Sigma^m_{j=1}(f^j_1 - f^j_{1P}))^\alpha} (f^j_1 - k^j_{1P})^\alpha + \\ & \delta \frac{(\Sigma^n_{i=1}(o^i_2 - o^i_{2P}))^\alpha}{(\Sigma^m_{j=1}(f^j_2 - f^j_{2P}))^\alpha} (f^j_2 - f^j_{2P})^\alpha \end{aligned} \tag{19}$$

However, since grain kept aside for production in the next period is redundant in the second period of the game we can say that $f^j_{2P} = 0$. Thus,

$$f^j_2 = (f^j_{1P})^\alpha \tag{20}$$

Thus from equations (19) and (20) we get

$$\begin{aligned} V^j = & max_{f^j_{1P}} \frac{(\Sigma^n_{i=1}(o^i_1 - o^i_{1P}))^\alpha}{(\Sigma^m_{j=1}(f^j_1 - f^j_{1P}))^\alpha} (f^j_1 - f^j_{1P})^\alpha + \\ & \delta \frac{((\Sigma^n_{i=1} o^i_1)^\beta)^\alpha}{((\Sigma^m_{j=1}(f^j_1)^\beta))^\alpha} ((f^j_{1P})^\beta)^\alpha \end{aligned} \tag{21}$$

Differentiating equation (21) with respect to f^j_{1P} we get,

$$\begin{aligned}
\frac{\partial V^j}{\partial f^j_{1P}} &= -\alpha \frac{(\Sigma^n_{i=1}(o^i_1 - o^i_{1P}))^\alpha}{(\Sigma^m_{j=1}(f^j_1 - f^j_{1P}))^\alpha}(f^j_1 - f^j_{1P})^{\alpha-1} \\
&+\alpha \frac{(\Sigma^n_{i=1}o^i_1 - o^i_{1P})^\alpha}{(\Sigma^m_{j=1}(f^j_1 - f^j_{1P}))^{\alpha+1}}(f^j_1 - f^j_{1P})^{\alpha} \\
&+\delta\alpha\beta \frac{(\Sigma^n_{i=1}(o^i_{1P}))^\alpha}{((\Sigma^m_{j=1}(f^j_{1P})^\beta))^{\alpha+1}}(f^j_{1P})^{\alpha\beta-1} \\
&-\delta\alpha\beta \frac{(\Sigma^n_{i=1}(o^i_{1P}))^\alpha}{((\Sigma^m_{j=1}(f^j_{1P})^\beta))^{\alpha+1}}(f^j_{1P})^{\alpha\beta+\beta-1} \qquad (22)
\end{aligned}$$

We need to check whether the agent is indeed maximizing her value function. We do this by differentiating equation (22) with respect to f^f_{1P} to get,

$$\begin{aligned}
\frac{(\partial V^j)^2}{\partial (f^j_{1P})^2} &= -\alpha(1-\alpha)(f^j_1 - f^j_{1P}))^{\alpha-2}\frac{(\Sigma^n_{i=1}(o^i_1 - o^i_{1P}))^\alpha}{(\Sigma^m_{j=1}(f^j_1 - f^j_{1P}))^\alpha} \\
&-\alpha^2(f^j_1)^{\alpha-1}\frac{(\Sigma^n_{i=1}(o^i_1 - o^i_{1P}))^\alpha}{(\Sigma^m_{j=1}(f^j_1 - f^j_{1P}))^{\alpha+1}} \\
&-\alpha^2(f^j_1)^{\alpha+1}\frac{(\Sigma^n_{i=1}(o^i_1 - o^i_{1P}))^\alpha}{(\Sigma^m_{j=1}(f^j_1 - f^j_{1P}))^{\alpha+1}} \\
&+\alpha(\alpha+1)(f^j_1 - f^j_{1P}))^{\alpha}\frac{(\Sigma^n_{i=1}(o^i_1 - o^i_{1P}))^\alpha}{(\Sigma^m_{j=1}(f^j_1 - f^j_{1P}))^{\alpha+2}} \\
&-\delta\alpha\beta(1-\alpha\beta)(f^j_{1P})^{\alpha\beta-2}\frac{(\Sigma^n_{i=1}o^i_{1P})^\alpha}{((\Sigma^m_{j=1}f^j_{1P})^\beta)^\alpha} \\
&-\delta(\alpha\beta)^2(f^j_{1P})^{\alpha\beta+\beta-2}\frac{(\Sigma^n_{i=1}o^i_{1P})^\alpha}{((\Sigma^m_{j=1}f^j_{1P})^\beta)^\alpha} \\
&-\delta(\alpha\beta)(\alpha\beta+\beta-1)(f^j_{1P})^{\alpha\beta+\beta-2}\frac{(\Sigma^n_{i=1}o^i_{1P})^\alpha}{((\Sigma^m_{j=1}f^j_{1P})^\beta)^{\alpha+1}} \\
&-\delta(\alpha\beta^2)(\alpha+1)(f^j_{1P})^{\alpha\beta+2\beta-2}\frac{(\Sigma^n_{i=1}o^i_{1P})^\alpha}{((\Sigma^m_{j=1}f^j_{1P})^\beta)^{\alpha+2}}
\end{aligned}$$

For the agent to be maximizing her utility we need $\frac{(\partial V^j)^2}{\partial (f^j_{1P})^2} < 0$. This implies that

$$\frac{1+\alpha}{\Sigma^m_{j=1}(f^j_1 - f^j_{1P})} < \frac{\alpha}{(f^j_1 - f^j_{1P})}$$

Assuming symmetry amongst the agents this translates into

$$1 < \alpha(m-1)$$

This is something that we need to check for when we solve for an equilibrium.

We would assume, given the set up of the model that there ought to be complementarity of action between the oil and foodgrain producers. Mathematically this would imply,

$$\frac{(\partial V^j)^2}{\partial f_{1P}^j o_{1P}^i} > 0$$

Differentiating equation (22) with respect to o_{1P}^i we get,

$$\begin{aligned}
\frac{(\partial V^j)^2}{\partial f_{1P}^j \partial o_{1P}^i} &= \alpha^2 (f_1^j - f_{1P}^j)^{\alpha-1} \frac{(\Sigma_{i=1}^n o_1^i - o_{1P}^i)^{\alpha-1}}{(\Sigma_{j=1}^m (f_1^j - f_{1P}^j))^{\alpha}} \\
&\quad - \alpha^2 (f_1^j - f_{1P}^j)^{\alpha} \frac{(\Sigma_{i=1}^n o_1^i - o_{1P}^i)^{\alpha-1}}{(\Sigma_{j=1}^m (f_1^j - f_{1P}^j))^{1+\alpha}} \\
&\quad + \alpha^2 \beta (f_{1P}^j)^{\alpha\beta-1} \frac{(\Sigma_{i=1}^n o_{1P}^i)^{\alpha-1}}{(\Sigma_{j=1}^m (f_{1P}^j)^{\beta})^{\alpha}} \\
&\quad - \alpha^2 \beta (f_{1P}^j)^{\alpha\beta+\beta-1} \frac{(\Sigma_{i=1}^n o_{1P}^i)^{\alpha-1}}{(\Sigma_{j=1}^m (f_{1P}^j)^{\beta})^{1+\alpha}}
\end{aligned}$$

From the above we see that $\frac{(\partial V^j)^2}{\partial f_{1P}^j o_{1P}^i} > 0$. This is exactly what we would expect. If we assume symmetry and were to obtain a similar result for the oil producers then we would have upward sloping reaction functions as the theory of supermodularity predicts. This would result in the possibility of obtaining multiple equilibria with regard to the aggregate quantities of oil and foodgrains produced in the first period. This added layer is completely missing in the partial equilibrium model on account of the reduction of the demand for oil to an inverse demand function that is downward sloping.

5.2.2 The oil producer's problem

The problem facing the oil producer is essentially that facing the grain producer. The sole difference being that the oil producer needs to decide how much oil she needs to leave in the ground for the next period. Thus, her production technology is essentially a storage technology. Let o^i_{tP} denote the stock of oil stored in the ground in period t and o^i_{tT} the amount of oil used for trade in period t. Since each agent of type i is endowed with an initial stock of oil, O^i_1 is given for each agent of type i. We put no restrictions on the initial endowment for the agents. The endowment that is put aside for 'production' is used for trade the next period. Thus we write:

$$O^i_t = o^i_{tP} + o^i_{tT}$$

The agent's budget constraint is given by

$$q_t o^i_{tT} = p_t f^i_{tC} \tag{23}$$

(where, p_t is the price of grain, q_t is the price of oil and f^i_{tC} is the amount of food consumed by the agent of type i in period t)

It must be that the value of the goods traded in each period must be equal. Thus we have

$$p_t \Sigma_j (f^j_t - f^j_{tP}) = q_t \Sigma_i (O^i_t - o^i_{tP}) \tag{24}$$

Each agent thus solves the following maximization problem,

$$V^i \quad = \quad max_{f^i_{1C}, f^i_{2C}} \Sigma^2_{t=1} \delta^{t-1} (f^i_{tC})^\alpha \tag{25}$$

$$V^i \quad = \quad max_{o^i_{1T}, o^i_{2T}} \Sigma^2_{t=1} \delta^{t-1} ((q_t/p_t) o^i_{1T})^\alpha$$

$$V^i \quad = \quad max_{o^i_{1T}, o^i_{2T}} ((q_1/p_1) o^i_{1T})^\alpha + \delta((q_2/p_2) o^i_{2T})^\alpha$$

Thus from equation (23), (24), and (25) we get

$$\begin{aligned} V^i \quad = \quad & max_{o^i_{1P}} \frac{(\Sigma^m_{j=1}(f^j_1 - f^j_{1P}))^\alpha}{(\Sigma^n_{i=1}(O^i_1 - o^i_{1P}))^\alpha} (O^i_1 - o^i_{1P})^\alpha \\ & + \delta \frac{(\Sigma^m_{j=1}(f^j_2 - k^j_{2P}))^\alpha}{(\Sigma^n_{i=1}(O^i_2 - o^i_{2P}))^\alpha} (O^i_2 - o^i_{2P})^\alpha \end{aligned} \tag{26}$$

However, since keeping oil aside in storage in the next period is redundant in the second period of the game, we can say that $o^i_{2P} = 0$. Thus,

$$O_2^i = o_{1P}^i \tag{27}$$

Thus from equations (26) and (27) we get

$$\begin{aligned} V^i &= max_{o_{1P}^i} \frac{(\Sigma_{j=1}^m (f_1^j - f_{1P}^j))^\alpha}{(\Sigma_{i=1}^n (O_1^i - o_{1P}^i))^\alpha} (O_1^i - o_{1P}^i)^\alpha \\ &+\delta \frac{(\Sigma_{j=1}^m (f_{1P}^j)^\beta)^\alpha}{(\Sigma_{i=1}^n (o_{1P}^i))^\alpha} (o_{1P}^i)^\alpha \end{aligned} \tag{28}$$

Differentiating equation (28) with respect to o_{1P}^i we get,

$$\begin{aligned} \frac{\partial V^i}{\partial o_{1P}^i} &= -\alpha \frac{(\Sigma_{j=1}^m (f_1^j - f_{1P}^j))^\alpha}{(\Sigma_{i=1}^n (O_1^i - o_{1P}^i)^\alpha} (O_1^i - o_{1P}^i)^{\alpha-1} \\ &+\alpha \frac{(\Sigma_{j=1}^m (f_1^j - f_{1P}^j))^\alpha}{(\Sigma_{i=1}^n (O_1^i - o_{1P}^i))^{1+\alpha}} (O_1^i - o_{1P}^i)^\alpha \\ &+\delta\alpha \frac{(\Sigma_{j=1}^m (f_{1P}^j)^\beta)^\alpha}{(\Sigma_{i=1}^n (o_{1P}^i))^\alpha} (o_{1P}^i)^{\alpha-1} \\ &-\delta\alpha \frac{(\Sigma_{j=1}^m (f_{1P}^j)^\beta))^\alpha}{(\Sigma_{i=1}^n (o_{1P}^i))^{1+\alpha}} (o_{1P}^i)^\alpha \end{aligned} \tag{29}$$

We would assume, given the set up of the model that there ought to be complementarity of action between the renewable and non renewable resource producers. Mathematically this would imply,

$$\frac{(\partial V^i)^2}{\partial f_{1P}^j o_{1P}^i} > 0$$

To check this we differentiate equation (29) with respect to f_{1P}^j to obtain,

$$\begin{aligned} \frac{(\partial V^i)^2}{\partial f_{1P}^j \partial o_{1P}^i} &= \alpha^2 (f_1^j - f_{1P}^j)^{\alpha-1} \frac{(\Sigma_{i=1}^n O_1^i - o_{1P}^i)^\alpha}{(\Sigma_{j=1}^m (O_1^i - o_{1P}^i))^{\alpha-1}} \\ &-\alpha^2 (O_1^i - o_{1P}^i)^\alpha \frac{(\Sigma_{i=1}^n f_1^j - f_{1P}^j)^{\alpha-1}}{(\Sigma_{i=1}^n (O_1^i - o_{1P}^i))^{1+\alpha}} \\ &+\delta\alpha^2 \beta (f_{1P}^j)^{\beta-1} \frac{(\Sigma_{i=1}^m (f_{1P}^i)^\beta)^{\alpha-1}}{(\Sigma_{i=1}^n (o_{1P}^i))^\alpha} \\ &-\delta\alpha^2 \beta (f_{1P}^j)^{\beta-1} \frac{(\Sigma_{j=1}^m (f_{1P}^j)^\beta)^{\alpha-1}}{(\Sigma_{i=1}^n (o_{1P}^i))^{1+\alpha}} \end{aligned} \tag{30}$$

For equation (30) to be greater than zero we require that $n > 1$. From the above we see that $\frac{(\partial V^j)^2}{\partial f^j_{1P} o^i_{1P}} > 0$. This is exactly what we would expect. Assuming symmetry, we would have upward sloping reaction functions as before just as the theory of supermodularity predicts. This would result in the possibility of obtaining multiple equilibria with regard to the aggregate quantities of oil and foodgrains produced in the first period.

There is another aspect to this issue. That is regarding the behavior of the asymmetrically endowed oil producers. This is the stylized fact that we hope to capture in our model, the theory of 'oil'igopoly. This would require that we check the reactions of the oil producers for a given level of renewable resource production. If the actions of the oil producers are strategic substitutes for each other then we have downward sloping reaction functions as show in the following sections. This translates mathematically into finding that $\frac{(\partial V^i)^2}{\partial o^i_{1P} \Sigma_{\neq i} o^i_{1P}} < 0$. This means that

$$\begin{aligned}
\frac{(\partial V^i)^2}{\partial o^i_{1P} \partial \Sigma_{\neq i} o^i_{1P}} &= -\alpha^2 (f^j_1 - k^j_{1P})^\alpha \frac{(\Sigma^n_{i=1} O^i_1 - o^i_{1P})^{\alpha-1}}{(\Sigma^m_{j=1}(O^i_1 - o^i_{1P}))^{\alpha+1}} \\
&+\alpha(1+\alpha)(O^i_1 - o^i_{1P})^\alpha \frac{(\Sigma^n_{i=1} f^j_1 - f^j_{1P})^\alpha}{(\Sigma^n_{i=1}(O^i_1 - o^i_{1P}))^{2+\alpha}} \\
&-\delta\alpha^2 (o^i_{1P})^{\alpha-1} \frac{(\Sigma^m_{j=1}(f^j_{1P})^\beta)^\alpha}{(\Sigma^n_{i=1}(o^i_{1P}))^{\alpha+1}} \\
&+\delta\alpha(1+\alpha)\beta(f^j_{1P})^{\beta-1} \frac{(\Sigma^m_{j=1}(f^j_{1P})^\beta)^{\alpha-1}}{(\Sigma^n_{i=1}(o^i_{1P}))^{1+\alpha}}
\end{aligned}$$

For this to be less than zero we require that

$$\frac{1+\alpha}{(\Sigma^n_{i=1}(o^i_1 - o^i_{1P}))} < \frac{\alpha}{(o^i_1 - o^i_{1P})}$$

Thus, as long as this condition is satisfied, we have downward sloping reaction functions. Then, appealing to theorems 3, 4, and 8 to 10, we can show that an equilibrium exists and provide conditions for when the equilibrium is unique.

5.3 An example

We sought to construct a fully strategic model that retained a key, though oft-neglected stylized fact regarding the production data of crude oil, namely, that countries with larger endowments of crude oil tend to produce a smaller proportion of their endowment than do countries with smaller endowments of crude oil. We believe that the fully strategic model provides a richer framework to understand the dynamics of oil production than the Cournot-Walras framework that is employed in the literature that largely rely on the assumption of a downward sloping linear demand function to drive their results, an assumption that does not find a firm basis in theory or empirical evidence.

First, let us look at a plot between the oil producers and the food producers. Let us assume symmetry for each of them. Consider the following parameter values:

$$m = 100 \quad n = 100 \quad \alpha = 0.7$$

$$\beta = 0.8 \quad \delta = 0.6 \quad f_{1P}^{f} = 100$$

$$o_{1P}^{0} = 100$$

(where m and n are the number of food and oil producers respectively.)

Using these values and the first order conditions we get Figure 2 below. The solid line indicates the reaction function of the oil producer and the dashed line indicates the reaction function of the foodgrain producer. As expected we see strategic complementarity between the actions of the oil and food producers. The reaction curves are upward sloping and they intersect at one stable equilibrium. We cannot rule out the possibility of multiple equilibria in the case of different parameter values.

Using the equilibrium values that we obtain from this run and the positing that there exist two types of oil producers, one with a bigger endowment of oil and the other with an smaller endowment of oil, we calculate the reaction functions and obtain the Figure 3 that shows the behavior of the two types of oil producers in equilibrium. The solid line indicates the reaction function of the bigger oil producer and the dashed line indicates the reaction function of the smaller oil producer.The figure clearly indicates that the oil producers with the bigger endowment consume a smaller fraction of the resource in the first period than the oil producers with a smaller endowment. This is in keeping with the theory of 'oil'igopoly. Since we have downward sloping reaction functions and the conditions of uniqueness are satisfied given the functional forms that we have assumed, we know that there should exist a stable, unique equilibrium which is observed in the Figure 3.

Figure 2

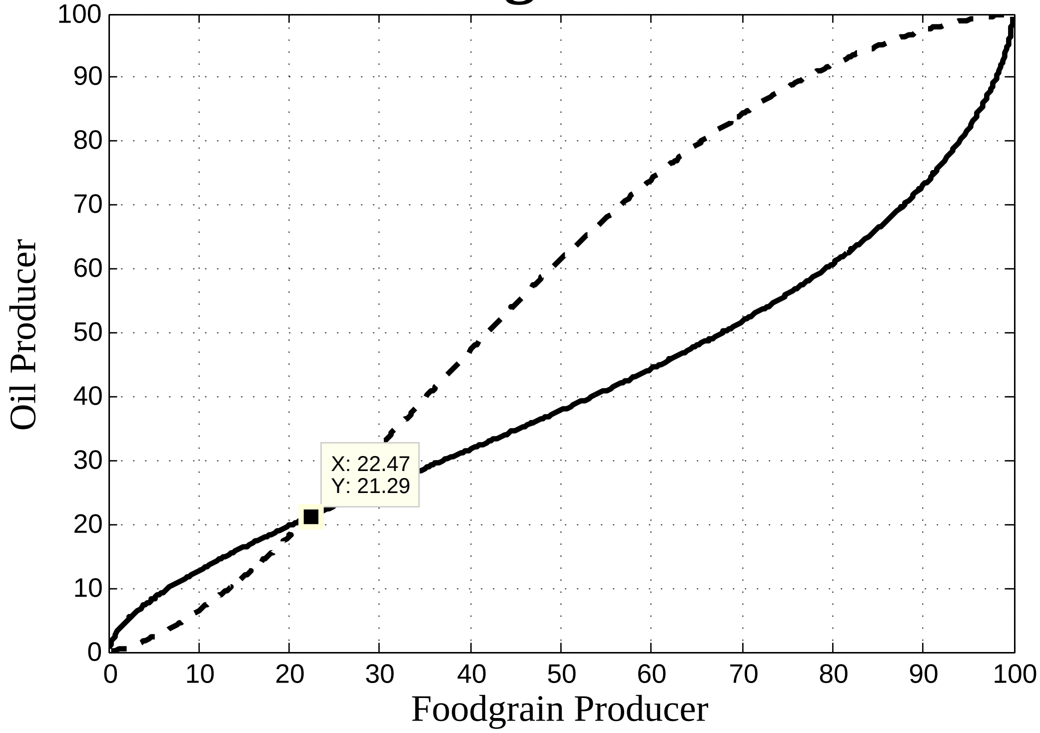
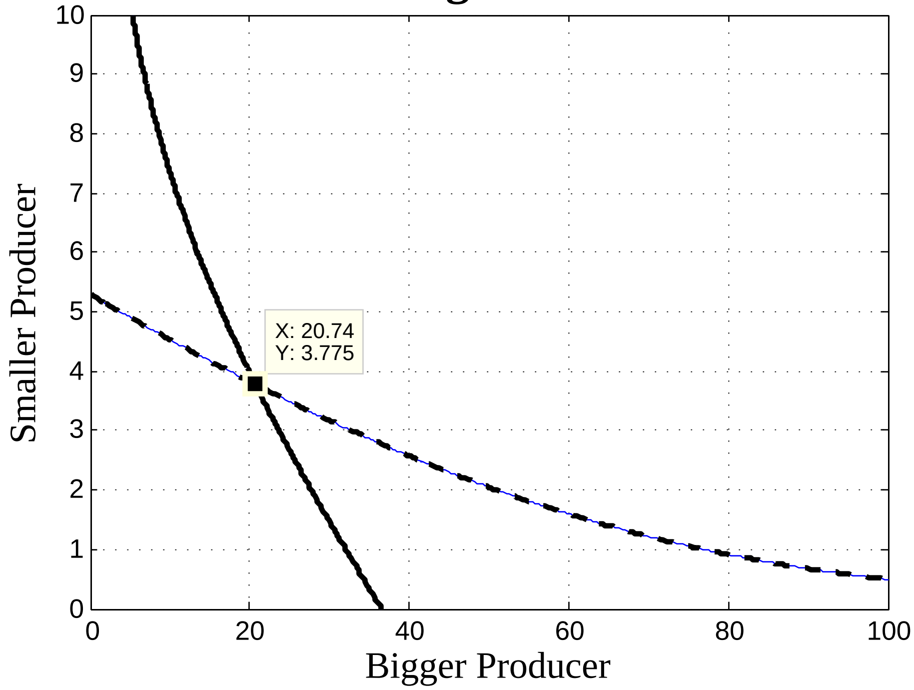

100
90
80
70
60
50
40
30
20
10
0
0
10
20
30
40
50
60
70
80
90
100
Oil Producer
Foodgrain Producer
X: 22.47
Y: 21.29

Figure 3


10
9
8
7
6
5
4
3
2
1
0
0
20
40
60
80
100
X: 20.74
Y: 3.775
Smaller Producer
Bigger Producer

6 Conclusion

In this paper we have considered two simultaneous-move, dynamic games that describe the behavior of oil producers. Both offer insight into the reserves-production relationship that one finds in the data. Indeed, they are in keeping with the prediction of the theory of 'oil'igopoly by Loury (1986) that oil producers endowed with large resource stocks produce a larger amount but a smaller percentage of their stocks than producers endowed with small resource stocks.

However, the data points to the existence of a deeper relationship when one considers individual countries. It would seem that the amount of resource produced by a country depends not only on its resource endowment but also the resource endowment and production decisions of other countries. The Cournot-Walras model is able to capture this relationship in addition to being able to match the original prediction of the theory. However, this is achieved at the expense of the assuming of a downward sloping inverse demand function for oil. This assumption is standard in the literature; however, there does not seem to be strong evidence for the same.

The fully strategic model dispenses with the assumption of a downward sloping inverse demand function by introducing trade between between strategic oil producers and strategic producers of the consumption good. The resource extraction problem has been studied with an emphasis on understanding the interactions between the oil producers. However, as the fully strategic model indicates, we might be missing the bigger picture wherein the interactions between oil and non-oil producers dictate the equilibrium attained. This simple, deterministic model, where the possibility of multiple equilibria engender coordination failure in the absence of a coordinating mechanism, could help improve our intuition as to the existence of oil price shocks, be they of the supply or demand kind.

An important aspect that is abstracted away in our models concerns the exploration decisions of oil producers. It would be instructive to see how this would alter production behavior. Furthermore, we have considered a simple, two-period case. Extending the model to the infinite horizon case would offer more insights. Most importantly answering questions regarding the relationship between reserves a country holds and the date of exhaustion of those reserves. This is our future direction.

7 Appendix

A detailed discussion of the concepts covered here can be found in Topkis (1998) and Vives (1999). A pair $(X, \leq)$, where X is a set and $\leq$ is a transitive, reflexive, antisymmetric binary relation, is a partially ordered set (poset); $(X, \leq)$ is totally ordered if, for all $x, y \in X$, $x \leq y$ or $y \leq x$ ($\leq$ is then a total order on X); $(X, \leq)$ is a lattice if whenever $x, y \in X$, both $x \wedge y = inf(x, y)$ and $x \vee y = sup(x, y)$ exist in X. Note that a totally ordered set is a lattice.

A nonempty subset A of X is a *sublattice* if for all $x, y \in A$, $x \wedge_X y$,$x \vee_X y \in A$, where $x \wedge_X y$ and $x \vee_X y$ are obtained taking the infimum and supremum as elements of X (as opposed to using the relative order on A). A lattice $(X, \leq)$ is complete if for every nonempty subset A of X, *infA*,*supA* exist in X. A nonempty subset A of X is subcomplete if $B \subseteq A$, $B \neq \emptyset$ implies $inf_X B$,$sup_X B \in A$, again taking *inf* and *sup* of B as a subset of X. For two subsets A, B of X, say that A is smaller than B in the strong set order if $a \in A, b \in B$ implies $a \wedge b \in A$, $a \vee b \in B$. Let $(X, \leq)$ be a lattice.

Say that a correspondence $\phi : X \to X$ is weakly increasing over $A \subseteq X$ if $x, y \in A$ and $x \leq y$ implies that there is $z \in \phi(x)$ and $z^{'} \in \phi(y)$ with $z \leq z^{'}$. Also, say that ϕ is increasing in the strong set order if $x \leq y$ implies that $\phi(x)$ is smaller in the strong set order than $\phi(y)$. Note that when ϕ is a function, i.e., single valued, both concepts coincide with the usual notion of "monotone weakly increasing." A correspondence $\phi : X \to X$ takes finite values if $\phi(x)$ is a finite set for all $x \in X$.

Let Z and Θ be subsets of $\Re^n$ and $\Re^l$ respectively. A function $f : Z \times \Theta \to \Re$ is said to be submodular in (z, θ) if it is the case that for all $x = (z, \theta)$ and $x^{'} = (z^{'}, \theta^{'})$ in $Z \times \Theta$ we have

$$f(x) + f(x^{'}) \geq f(x \vee x^{'}) + f(x \wedge x^{'})$$

A function $f : Z \times \Theta \to \Re$ is said to satisfy decreasing differences in (z, θ) if for all pairs (z, θ) and $(z^{'}, \theta^{'})$ in $Z \times \Theta$, it is the case that $z \leq z^{'}$ and $\theta \leq \theta^{'}$ implies

$$f(z^{'}, \theta) - f(z, \theta) \geq f(z^{'}, \theta^{'}) - f(z, \theta^{'})$$

Let Z and Θ be subsets of $\Re^n$ and $\Re^l$ respectively. A function $f : Z \times \Theta \to \Re$

is said to be supermodular in (z, θ) if it is the case that for all $x = (z, \theta)$ and $x^{'} = (z^{'}, \theta^{'})$ in $S \times \Theta$ we have

$$f(x) + f(x^{'}) \leq f(x \vee x^{'}) + f(x \wedge x^{'})$$

A function $f : Z \times \Theta \rightarrow \Re$ is said to satisfy increasing differences in (z, θ) if for all pairs (z, θ) and $(z^{'}, \theta^{'})$ in $Z \times \Theta$, it is the case that $z \leq z^{'}$ and $\theta \leq \theta^{'}$ implies

$$f(z^{'}, \theta) - f(z, \theta) \geq f(z^{'}, \theta^{'}) - f(z, \theta^{'})$$

Theorem 8 *A function $f : Z \times \Theta \rightarrow \Re$ is submodular if and only if f has decreasing differences on Z*

Proof. Let $Z \subset \Re^m$. For $z \in Z$, we will denote by $(z_{-ij}, z_i^{'}, z_j^{'})$ the vector z, but with z_i and z_j replaced by $z_i^{'}$ and $z_j^{'}$.Suppose, without loss in generality, that $z_i^{'} \geq z_i$ and $z_j^{'} \geq z_i$. Suppose that $z = (z_1, ..., z_m)$ and $z^{'} = (z_1, ..., z_i^{'}, z_j^{'}, ..., z_m)$. Let $w = (z_1, ..., z_i^{'}, z_j, ..., z_m)$ and $w^{'} = (z_1, ..., z_i, z_j^{'}, ..., z_m)$.Thus we have,

$$w \wedge w^{'} = z^{'}$$
$$w \vee w^{'} = z$$

Since f is submodular on z we have

$$f(z; \theta) + f(z^{'}; \theta) \geq f(z \vee z^{'}; \theta) + f(z \wedge z^{'}; \theta)$$

Rearranging and using the notation for w and $w^{'}$ we get,

$$f(z_1, ..., z_i^{'}, z_j^{'}, ..., z_m; \theta) + f(z_1, ..., z_i, z_j^{'}, ..., z_m; \theta) \geq$$
$$f(z_1, ..., z_i^{'}, z_j, ..., z_m; \theta) + f(z_1, ..., z_i, z_j, ..., z_m; \theta)$$

To see that decreasing differences on Z implies submodularity on Z, consider any z and $z^{'}$ in Z. We are required to show that

$$f(z; \theta) + f(z^{'}; \theta) \geq f(z \vee z^{'}; \theta) + f(z \wedge z^{'}; \theta)$$

If $z \geq z^{'}$ or $z \leq z^{'}$, this inequality trivially holds. So suppose that z and $z^{'}$ are not comparable under $\leq$. For notational convenience, arrange the coordinates of z and $z^{'}$ so that

$$z \vee z^{'} = (z_1^{'}, ..., z_k^{'}, z_{k+1}, ..., z_m)$$

$$z \wedge z^{'} = (z_1, ..., z_k, z_{k+1}^{'}, ..., z_m^{'})$$

As z and $z^{'}$ are not comparable under $\leq$, we must have $0 < k < m$. Now for $0 \leq i \leq j \leq m$, define

$$z^{i,j} = (z_1^{'}, ..., z_i^{'}, z_{i+1}^{'}, ..., z_j, z_{j+1}^{'}, ..., z_m^{'})$$

Then we have,

$$\begin{aligned} z^{0,k} &= z \wedge z^{'} \\ z^{k,m} &= z \vee z^{'} \\ z^{0,m} &= z \\ z^{k,k} &= z^{'}. \end{aligned}$$

Since f has decreasing differences on Z, it must be that for all $0 \leq i < k \leq j < m$

$$f(z^{i+1,j+1}; \theta) - f(z^{i,j+1}; \theta) \leq$$
$$f(z^{i+1,j}; \theta) + f(z^{i,j}; \theta)$$

Thus we have for $k \leq j < m$

$$\begin{aligned} f(z^{k,j+1}; \theta) - f(z^{0,j+1}; \theta) &= \Sigma_{i=0}^{k-1}[f(z^{i+1,j+1}; \theta) - f(z^{i,j+1}; \theta)] \\ &\leq \Sigma_{i=0}^{k-1} f(z^{i+1,j}; \theta) + f(z^{i,j}; \theta) \\ &= f(z^{k,j}; \theta) - f(z^{0,j}; \theta). \end{aligned}$$

As this inequality holds for all j satisfying $k \leq j < m$. it follows that the left-hand side is takes its highest value when $j = m - 1$, while the right-hand side takes its lowest value when $j = k$. Therefore,

$$f(z^{k,m}; \theta) - f(z^{0,m}; \theta) \leq f(z^{k,k}; \theta) + f(z^{0,k}; \theta)$$

Which is exactly,

$$f(z;\theta) + f(z^{'};\theta) \geq f(z \vee z^{'};\theta) + f(z \wedge z^{'};\theta)$$

Since z and $z^{'}$ were chosen arbitrarily, we have shown that f is submodular on Z.

■

Theorem 9 *Let Z be an open sublattice of $\Re^m$ for a given θ. A C^2 function $f : Z \times \Theta \rightarrow \Re$ is submodular on Z for a given θ if and only if for all $z \in Z$, we have*

$$\frac{\partial^2 f(z;\theta)}{z_i z_j} \leq 0$$

(for all $i, j = 1, ..., m, i \neq j$).

Proof. Let f be a C^2 function on $Z \subset \Re^m$ for a given $\theta \in \Theta$. By theorem 2, f is submodular on Z for a given $\theta \in \Theta$ if and only if for all $z \in Z$, for all distinct i and j, and for all $\epsilon > 0$ and $\delta > 0$, we have,

$$f(z_{-ij}, z_i + \epsilon, z_j + \delta; \theta) - f(z_{-ij}, z_i + \epsilon, z_j; \theta) \quad \leq \quad f(z_{-ij}, z_i, z_j + \delta; \theta) + f(z_{-ij}, z_i, z_j; \theta)$$

Dividing both sides the positive quantity δ and letting $\delta \rightarrow 0$, we see that f is submodular on Z if and only if for all $z \in Z$, for all distinct i and j, and for all $\epsilon > 0$,

$$\frac{\partial f(z_{-ij}, z_i + \epsilon, z_j; \theta)}{\partial z_j} \quad \leq \quad \frac{\partial f(z_{-ij}, z_i, z_j; \theta)}{\partial z_j}$$

Subtracting the right-hand side from the left-hand side, dividing by the positive quantity ϵ, and letting $\epsilon \rightarrow 0$, f is seen to be submodular on Z for a given value of θ if and only if for all $z \in Z$, and for all distinct i and j, we have

$$\frac{\partial^2 f(z;\theta)}{z_i z_j} \leq 0$$

(for all $i, j = 1, ..., m, i \neq j$).

■

Theorem 10 *Let $Z = S \times T$. Let S be a compact sublattice on $\Re_+$, let T be a sublattice of $\Re^+_{n-1}$ and let Θ be a sublattice on $\Re^k_+$ and $f : Z \times \Theta \rightarrow \Re_+$ be a continuous function on $Z = S \times T$ for each fixed $t \in T$ and $\theta \in \Theta$. Suppose* *that f is submodular on Z for each fixed $\theta \in \Theta$. Let the correspondence ρ from T to S be defined as*

$$\rho^* = argmax f(s,t)|x \in S$$

1. For each $t \in T$, $\rho(t;\theta)$ is a nonempty, compact poset of $\Re_+$ and admits a greatest element.

2. $\rho^*(t^{'};\theta) \geq \rho^*(t;\theta)$ for all $t^{'} \leq t$

Proof. Since f is continuous on S for all $t \in T$ and S is compact, $\rho^*(t;\theta)$ is nonempty for all $t \in T$. Fix t and consider a sequence s_n in $\rho^*(t;\theta)$ converging to an $s \in S$. Then, for any $s^{'} \in S$, we have $f((s_n,t),\theta) \geq f((s,t),\theta)$. By continuity of $f((\bullet,t),\theta)$ we have $f((s,t),\theta) \geq f((s^{'},t),\theta)$. This implies that $s \in \rho^*(t;\theta)$. Thus, $\rho^*(t;\theta)$ is closed and compact as S is compact. Furthermore, the meet and the join of any two elements in $\rho^*(t;\theta)$ is a sublattice that admits a greatest element. This completes the proof of part 1.

Now consider t and $t^{'}$ such that $t \geq t'$. Let $s \in \rho^*(t;\theta)$ and $s^{'} \in \rho^*(t^{'};\theta)$. By optimality we have

$$f((s,t);\theta) - f((s^{'},t);\theta) \quad \geq \quad f((s,t^{'});\theta) + f((s^{'},t^{'});\theta)$$

Since f is submodular in $Z = S \times T$, we know from Theorem 2 that it would display decreasing differences in X. Suppose that $s \geq s^{'}$, then

$$f((s,t);\theta) - f((s^{'},t);\theta) \quad \leq \quad f((s,t^{'});\theta) + f((s^{'},t^{'});\theta)$$

From these two equations we have a contradiction.This means that $s \leq s^{'}$. Therefore, we have $\rho^*(t^{'};\theta) \geq \rho^*(t;\theta)$ for all $t^{'} \leq t$. ∎

References

[1] Cordella, T and Datta, M. (2002), Intertemporal Cournot and Walras equilibria: An illustration, International Economic Review; 43(1): pp.137-153.

[2] Folmer, H and Pierre von Mouche, M. (2004), On a less known Nash equlibrium uniqueness result, Journal of Mathematical Sociology; 28: pp.67-80.

[3] Gabay, D. and H. Moulin (1980), On the uniqeness and stability of Nash equilibrium in non cooperative games, Applied stochastic control in eonometric and management ecience, Amsterdam: North-Holland Publishing Company.

[4] Grandmont, J. (1993), Behavioural Heterogeneity and Cournot oiligopoly equilibrium, Ricerche Economiche, 47: pp.167-187.

[5] Johansen, S. and K. Juseliu (1990), Maximum Likelihood Estimation and the Inference on Cointegration-with applications to the demand for money, Oxford Bulletin for Economics and Statistics; 52, pp. 169-210.

[6] Kukushkin, N. (1994), A fixed point theorem for decreasing mappings, Economics Letters; 46: pp.23-26.

[7] Loury, G (1986), The theory of 'oil'igopoly: Cournot equilibrium in exhaustible resources with fixed supplies, International Economic Review; 27: pp.285-301.

[8] Pickering, A (2008), The oil reserves production ratio, Energy Journal; 30: pp.352-370.

[9] Polasky, S (1992), Do oil producers act as 'oil'igopolists?, Journal of Environmental Economics and Management; 23: pp.216-47.

[10] Reinganum, J. F. and N. L. Stokey (1985), Oligopoly extraction of a common property natural resource: The importance of the period of commitment in dynamic games, International Economic Review; 26: pp.161-173

[11] Roy, S. and T. Sabarwal (2008), On the non-lattice structure of the equilibrium set in games with strategic substitutes, Economic Theory; 37(1), pp. 161-169.

[12] Topkis, M. (1998), Supermodularity and complementarity, Princeton, NJ: Princeton University Press.

[13] Vives, X. (1999), Oiligopoly Pricing: Old ideas and new tools, MA: MIT Press.